

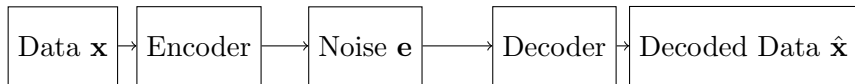
Coding Theory: Linear Codes and their Dual

A Generic Communication Channel

Transmitter

Channel

Receiver



$$\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{F}_q^k \mapsto \mathbf{c} = \underbrace{(c_1, \dots, c_n)}_{\text{codeword, } n \geq k} = (x_1, \dots, x_k) \underbrace{[\mathbf{I}_k | A]}_{\substack{\text{systematic} \\ \text{generator matrix}}} \in \mathbb{F}_q^n$$

Generator matrix

$$\mathcal{C} = \{\mathbf{c} = \mathbf{x}G, \mathbf{x} \in \mathbb{F}_q^k\}$$

Parity check matrix

$$\mathcal{C} = \{\mathbf{v} \in \mathbb{F}_q^n, H\mathbf{v}^T = \mathbf{0}\}$$

Linear Codes

■ Parity check matrices

The (7, 4) Hamming code

- Dimension: $k = 4$.
- Length: $n = 7$.
- Alphabet: \mathbb{F}_2 .
- Encoding given by the generator matrix $G = [\mathbf{I}_4 | A]$, with corresponding parity check matrix $H = [-A^T | \mathbf{I}_{n-k}]$:

$$G = \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array} \right]$$

Linear Codes

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Dual code \mathcal{C}^\perp

Given an (n, k) linear code \mathcal{C} over \mathbb{F}_q , \mathcal{C}^\perp is the $(n, n - k)$ linear code generated by the rows of its parity check matrix H .

To have a generator matrix, we simply need a matrix whose rows are independent, they then span the code.

The rows of H ($H = [-A^T | \mathbf{I}_{n-k}]$ in systematic form) are independent.

Linear Codes

■ Parity check matrices

For \mathcal{C}

\mathcal{C} is generated by k basis vectors in \mathbb{F}_q^n , placed as rows in G .

\mathcal{C} is the kernel of some H , that is $HG^T \mathbf{x}^T = \mathbf{0}$ for all $\mathbf{x} \in \mathbb{F}_q^k$ and $HG^T = \mathbf{0}$.

For \mathcal{C}^\perp

\mathcal{C}^\perp is generated by $n - k$ basis vectors in \mathbb{F}_q^n , placed as rows in H .

\mathcal{C}^\perp is the kernel of some \tilde{G} , that is $\tilde{G}H^T \mathbf{x}^T = \mathbf{0}$ for all $\mathbf{x} \in \mathbb{F}_q^{n-k}$ and $\tilde{G}H^T = \mathbf{0}$.

Linear Codes

■ Parity check matrices

For \mathcal{C}

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For \mathcal{C}^\perp

\mathcal{C}^\perp is generated by $n - k$ basis vectors in \mathbb{F}_q^n , placed as rows in H .

\mathcal{C}^\perp is the kernel of some \tilde{G} , that is $\tilde{G}H^T \mathbf{x}^T = \mathbf{0}$ for all $\mathbf{x} \in \mathbb{F}_q^{n-k}$ and $\tilde{G}H^T = \mathbf{0}$.

$$HG^T = \mathbf{0}, \quad H\tilde{G}^T = \mathbf{0} \xrightarrow[\text{dimension}]{\text{kernel}} \tilde{G} = G.$$

Dual code \mathcal{C}^\perp

Given an (n, k) linear code \mathcal{C} over \mathbb{F}_q ,
 $\mathcal{C}^\perp = \{\mathbf{v} \in \mathbb{F}_q^n, \mathbf{c} \cdot \mathbf{v}^T = \mathbf{0} \text{ for all } \mathbf{c} \in \mathcal{C}\}.$

The usual inner product of vectors $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ applies in \mathbb{F}_q^n :

$$\mathbf{x} \cdot \mathbf{y}^T = \sum_{i=1}^n x_i y_i.$$

Linear Codes

■ Dual codes

We know:

$$\begin{aligned}\mathcal{C}^\perp &= \{\mathbf{v} \in \mathbb{F}_q^n, \mathbf{c} \cdot \mathbf{v}^T = \mathbf{0} \text{ for all } \mathbf{c} \in \mathcal{C}\} \\ &= \{\mathbf{v} \in \mathbb{F}_q^n, \mathbf{x}G \cdot \mathbf{v}^T = \mathbf{0} \text{ for all } \mathbf{x} \in \mathbb{F}_q^k\}\end{aligned}$$

Let \mathbf{g}_i , $i = 1, \dots, k$ be the rows of G . Then $0 = \sum_{i=1}^k x_i \mathbf{g}_i \cdot \mathbf{v}^T$ for any x_i implies $0 = \mathbf{g}_i \cdot \mathbf{v}^T$ for every i and

$$\mathcal{C}^\perp = \{\mathbf{v} \in \mathbb{F}_q^n, G\mathbf{v}^T = \mathbf{0}\}.$$

Linear Codes

■ Dual codes

From \mathcal{C} to \mathcal{C}^\perp

Generator matrix of \mathcal{C} : G

Parity check matrix of \mathcal{C}^\perp :

G

Generator matrix of \mathcal{C}^\perp : H

From \mathcal{C}^\perp to \mathcal{C}

Generator matrix of \mathcal{C}^\perp : H

Parity check matrix of $(\mathcal{C}^\perp)^\perp$:

H

Generator matrix of $(\mathcal{C}^\perp)^\perp$: G

Linear Codes

■ Dual codes

From \mathcal{C} to \mathcal{C}^\perp

Generator matrix of \mathcal{C} : G

Parity check matrix of \mathcal{C}^\perp :

G

Generator matrix of \mathcal{C}^\perp : H

From \mathcal{C}^\perp to \mathcal{C}

Generator matrix of \mathcal{C}^\perp : H

Parity check matrix of $(\mathcal{C}^\perp)^\perp$:

H

Generator matrix of $(\mathcal{C}^\perp)^\perp$: G

- Both definitions of dual are equivalent.
- The dual of \mathcal{C}^\perp is \mathcal{C} : $(\mathcal{C}^\perp)^\perp = \mathcal{C}$.

Linear Codes

■ Dual codes

Repetition and single parity check codes

A generator matrix (in systematic form) of the repetition code:

$$[1|1 \dots, 1].$$

A parity check matrix (in systematic form) over \mathbb{F}_2 :

$$\left[\begin{array}{c|c} 1 & \\ \vdots & \mathbf{I}_{n-k} \\ 1 & \end{array} \right] \left. \vphantom{\begin{array}{c|c} 1 & \\ \vdots & \mathbf{I}_{n-k} \\ 1 & \end{array}} \right\} \begin{array}{l} \text{generator matrix} \\ \text{for the single parity} \\ \text{check code} \end{array}$$

A self-orthogonal code \mathcal{C}

A code \mathcal{C} which is
included in its dual:
 $\mathcal{C} \subseteq \mathcal{C}^\perp$

The $(n, 1)$ repetition code over \mathbb{F}_2 is self-orthogonal if n even.

To have $(n - 1) \equiv 1 \pmod{2}$, we need n even. Then $c_n = \sum_{i=1}^{n-1} x_i$ for both $(0, \dots, 0)$ and $(1, \dots, 1)$, and they are in the single parity check code.

A self-dual code \mathcal{C}

A code \mathcal{C} which is equal to its dual: $\mathcal{C} = \mathcal{C}^\perp$

The $(4, 2)$ code over \mathbb{F}_3 (called **tetracode**) given by the generator matrix

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

is self-dual.

Linear Codes

■ Dual codes

The (4,2) tetracode over \mathbb{F}_3

\mathcal{C}^\perp has generator matrix H

$$G = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}, \quad H = \begin{bmatrix} -1 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

Rewrite H in systematic form:

$$\begin{aligned} H &\rightarrow \begin{bmatrix} -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix} = G \end{aligned}$$

Linear Codes

■ Dual codes

The (4,2) tetracode over \mathbb{F}_3

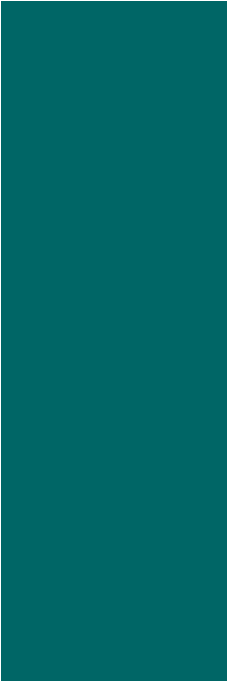
$$\mathcal{C}^\perp = \{\mathbf{v} \in \mathbb{F}_q^n, \mathbf{c} \cdot \mathbf{v}^T = \mathbf{0} \text{ for all } \mathbf{c} \in \mathcal{C}\}.$$

$$G = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}, \mathbf{c} = (x_1, x_2, x_1 + x_2, x_1 - x_2)$$

Then for $\mathbf{v} = (v_1, v_2, v_3, v_4) \in \mathbb{F}_3^4$, we want:

$$(v_1, v_2, v_3, v_4) \cdot \mathbf{c}^T = v_1x_1 + v_2x_2 + v_3(x_1 + x_2) + v_4(x_1 - x_2) = 0.$$

This means $v_1 + v_3 + v_4 = 0$ and $v_2 + v_3 - v_4 = 0$. Solve to find $v_3 = v_2 + v_1$ and $v_4 = v_1 - v_2$.



Two definitions of a dual code
self-orthogonal code
self-dual code