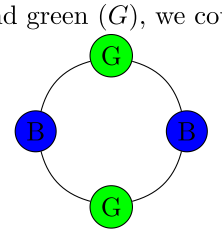
Chapter 1

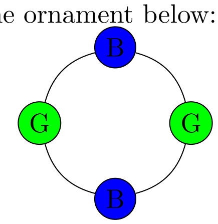
Pólya's Enumeration Theorem

1.1 Counting Necklaces

Consider a decorative ornament that consists of n coloured beads, arranged on a circular loop of strings. This can be represented as a word of length n over a suitable alphabet of colours. For example, for $n = 4$, and the colours blue (B) and green (G), we could have $GBGB$:



We could also label the above ornament $BGBG$, and similarly label $GBGB$ the ornament below:

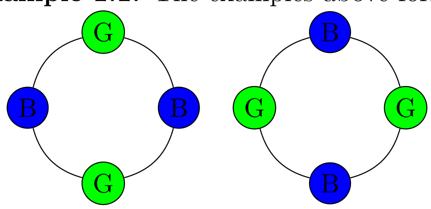


This is because the second ornament is drawn as a rotated version of the first ornament, but they are in fact the same ornament. The labels should thus correspond to the same word, and we say that two words that differ uniquely by a rotation of letters represent the same ornament, and they are called *equivalent*: $GBGB \equiv BGBG$. One may easily check that this indeed defines an equivalence relation (take the identity rotation for reflexivity, the reverse rotation for symmetry, and the combination of rotations for transitivity).

Definition 1.1. An (n, k)-*necklace* is an equivalence class of words of length n

over an alphabet of size k, under rotation.

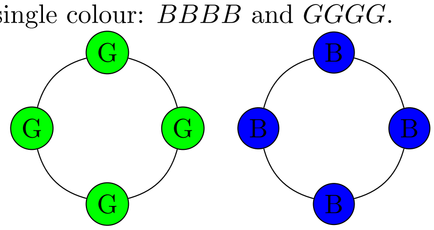
Example 1.1. The examples above form a $(4, 2)$-necklace:



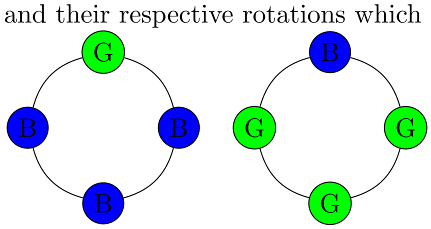
As usual when dealing with equivalent classes, one picks one representative per class.

Problem 1. [Necklace Enumeration Problem] Given n and k, how many (n, k)-necklaces are there?

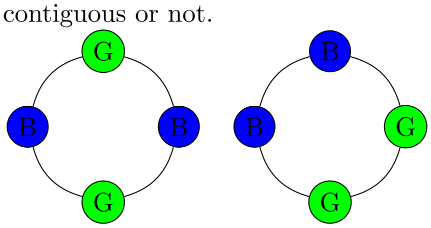
Example 1.2. Suppose $n = 4$ and $k = 2$ as above. Let us try to count how many necklaces with 4 beads and two colours there are. We have necklaces with a single colour: $BBBB$ and $GGGG$.



Then we have necklaces with only one bead B, and those with only one bead G, and their respective rotations which are not counted as different necklaces:



Then we have necklaces with exactly two beads of each colour, which could be contiguous or not.



This gives us a total of 6 necklaces.

This was easy by hand. Suppose now we have $n = 6$ beads, but $k = 5$ colours, what would be the answer? It is probably not advised to try this case by hand, since the claim is that there are 2635 such necklaces. We will develop next the tools to be able to prove this.

The problem involves two objects of different natures. The set X of words of length n over an alphabet of size k only has a set structure, it has size $|X| = k^n$. The set of n rotations (or rotations of angle $2\pi s/n$, $s = 0, \dots, n-1$) has the structure of (is isomorphic to) the cyclic group of order n, which we denote by C_n.

How the group structure of rotations gives structure to the set X is formally captured by the notion of group action on a set.

Definition 1.2. A (left) group action φ of a group G on X is a function

$$\varphi : G \times X \to X,\ (g, x) \mapsto \varphi(g, x) = g * x$$

that satisfies:

- **identity.** $1 * x = x$ for all $x \in X$.
- **compatibility.** $(gh) * x = g * (h * x)$ for all $g, h \in G$, for all $x \in X$.

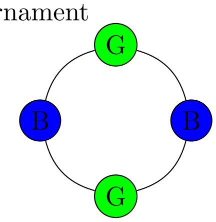
We can check here that our necklace setting does fit the framework of group action: 1 is the rotation that does not do anything on a word, so that the identity property is satisfied. As for compatibility, if we compose two rotations first, and apply the result on a word, it does give the same thing as applying the first rotation, and then applying the second one.

Definition 1.3. For a group G acting on a set X, the orbit $\mathrm{Orb}(x)$ of $x \in X$ is by definition

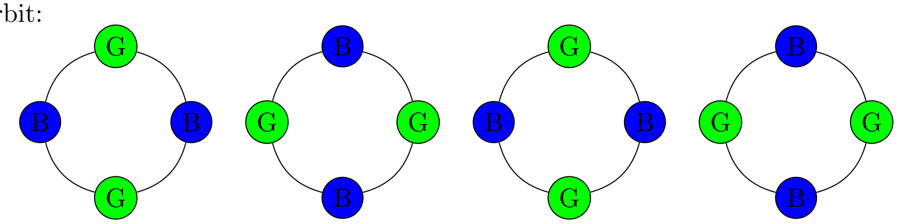
$$\mathrm{Orb}(x) = \{g * x,\ g \in G\}.$$

In terms of necklaces, $x \in X$ is a word of length n over an alphabet of size k, and g are rotations. An orbit for x is thus obtained by taking the chosen word x and applying on it all possible rotations in C_n.

Example 1.3. For $n = 4$ and $k = 2$, we have 4 rotations (by $\pi/2$, π, $3\pi/2$ and the identity), this is isomorphic to the cyclic group C_4. Then consider the ornament



and apply the 4 rotations, starting from the identity, to get the following orbit:



which thus consists of two distinct colourings.

The set of orbits is usually denoted by X/G:

$$X/G = \{\mathrm{Orb}(x),\ x \in X\}.$$

It is useful to notice that orbits partition X (and in that, the group action of G on X does tell us something about the structure of X).

Lemma 1.1. *Orbits under the action of the group G partition the set X.*

Proof. Firstly, the union of orbits is actually the whole of X:

$$\bigcup_{x \in X} \mathrm{Orb}(x) = X.$$

This is happening because since G is a group, it contains an identity element 1, so $1 * x \in \mathrm{Orb}(x)$, then the "identity" property of the group action implies that $x \in \mathrm{Orb}(x)$ for every orbit. Next two orbits $\mathrm{Orb}(x), \mathrm{Orb}(y)$ are either disjoint, or they are the same. Suppose that they are not disjoint, then there exists an element z that lives in both the orbits $\mathrm{Orb}(x)$ and $\mathrm{Orb}(y)$, then

$$z = g * x = g' * y \Rightarrow x = g^{-1}g' * y \in \mathrm{Orb}(y).$$

We note that we used twice group axioms, once to invert g, and once to say that $g^{-1}g' \in G$. We just showed that $x \in \mathrm{Orb}(y)$ and thus $\mathrm{Orb}(x) \subseteq \mathrm{Orb}(y)$. By repeating the same argument, we show the reverse inclusion. □

Based on what we just defined, we can rephrase Problem 1:

Problem 2. [Necklace Enumeration Problem] Given n and k, how many orbits of X under the action of C_n are there?

At this point, we may wonder why it was worth the effort to take our counting necklace problem and translate it into a problem of counting orbits under a group action, which does not seem an easier formulation (at first). The point of the reformulation is the result called *Burnside Lemma* (even though it was not proven by Burnside, so other authors call it Cauchy Frobenius Lemma).

Proposition 1.2. [Burnside Lemma] *Let G be a finite group action on a set X. Then*

$$|X/G| = \frac{1}{|G|}\sum_{g \in G} |\mathrm{Fix}(g)|,\ \mathrm{Fix}(g) = \{x \in X,\ g * x = x\}.$$

Before giving a proof of this result, let us see how useful it is for our necklace problem.

Example 1.4. Suppose $n = 6$ and $k = 2$. We could of course follow the same approach as in Example 1.2, that is, list ornaments based on how many beads are of the same colour:

- $BBBBBB$ and $GGGGGG$,

Figure 1.1: William Burnside (1852-1927)

- $GBBBBB$ and $BGGGGG$,
- $GGBBBB$, $GBGBBB$, $GBBGBB$, and the same pattern with reversed colours, $BBGGGG$, $BGBGGG$, $BGGBGG$,
- $GGGBBB$, $GGBGBB$, $GGBBGB$, $GBGBGB$ (note that the reversed colours do not give anything new up to rotation).

This list looks ok, but how do we make sure we got the right list? A first simple observation is that for $n = 4$, we had only $2^4 = 16$ possible words to check, in this case, we would have $2^6 = 64$ possible words, if we want to check them all, then we need to make sure we remove all the words equivalent up to rotation. So let us try Burnside Lemma. The group action on X is C_6, it has a generator g, which in cycle notation is $g = (1, 2, 3, 4, 5, 6)$. Then

$$\begin{aligned}
g^2 &= (135)(246) \\
g^3 &= (14)(25)(36) \\
g^4 &= (153)(264) \\
g^5 &= (165432) \\
g^6 &= (1)(2)(3)(4)(5)(6)
\end{aligned}$$

and we need to compute $\text{Fix}(g^i)$ for each i, that is we want ornaments which are invariant under rotation by g^i. Now g fixes only 2 words, $BBBBBB$ and $GGGGGG$, so $|\text{Fix}(g)| = 2$. Then g^2 fixes words with the same color in position 1,3,5 and in position 2,4,6, these are $BBBBBB$, $GGGGGG$, $BGBGBG$ and $GBGBGB$ (yes, the last two are obtained by rotation of each other, but remember that there is also an average by the number of elements of the group in the final formula), so $|\text{Fix}(g^2)| = 4$. We observe in fact that within one cycle, all the beads have to be of the same color, thus what matters is the number of cycles. Once this observation is made, we can easily compute:

$$\begin{array}{rcl}
g = & (123456) & |\mathrm{Fix}(g)| = 2^1 \\
g^2 = & (135)(246) & |\mathrm{Fix}(g^2)| = 2^2 \\
g^3 = & (14)(25)(36) & |\mathrm{Fix}(g^3)| = 2^3 \\
g^4 = & (153)(264) & |\mathrm{Fix}(g^4)| = 2^2 \\
g^5 = & (165432) & |\mathrm{Fix}(g^5)| = 2^1 \\
g^6 = & (1)(2)(3)(4)(5)(6) & |\mathrm{Fix}(g^6)| = 2^6
\end{array}$$

and we see that the number of necklaces is

$$\frac{1}{6}(2 + 2^2 + 2^3 + 2^2 + 2 + 2^6) = 14.$$

This does not really tell whether our above list was correct, but this shows that we got the right number of necklaces.

The above example shows that the number k of colours does not play a role but for being the basis of the exponents, so for $n = 6$ beads in general, we have

$$\begin{array}{rcl}
g = & (123456) & |\mathrm{Fix}(g)| = k \\
g^2 = & (135)(246) & |\mathrm{Fix}(g^2)| = k^2 \\
g^3 = & (14)(25)(36) & |\mathrm{Fix}(g^3)| = k^3 \\
g^4 = & (153)(264) & |\mathrm{Fix}(g^4)| = k^2 \\
g^5 = & (165432) & |\mathrm{Fix}(g^5)| = k \\
g^6 = & (1)(2)(3)(4)(5)(6) & |\mathrm{Fix}(g^6)| = k^6
\end{array}$$

and we see that the number of necklaces is

$$\frac{1}{6}(2k + 2k^2 + k^3 + k^6).$$

We can thus set $k = 5$ in this formula to obtain the number of necklaces with 5 colours and 6 beads, if we want to give an answer to the question asked after Example 1.2.

This formula for counting necklaces with 6 beads already shows why the formulation in terms of group action was a good idea. Thanks to Burnside Lemma, it becomes easy to compute a quantity which grows pretty quickly as a function of k.

Now this formula supposes that $n = 6$, and we would like to have a general formula, that is a formula valid for an arbitrary n. We saw above that the number of words fixed by an element $g \in C_n$ is determined by the number of cycles in its cycle decomposition: if g has $c(g)$ cycles, then it fixes $k^{c(g)}$ words, and the number of (n, k)-necklaces is

$$\frac{1}{n} \sum_{g \in C_n} k^{c(g)}.$$

Let us thus try to understand $c(g^m)$ for g^m a rotation of angle $2\pi m/n$. Such a rotation, in cycle notation, will look like

$$(i, i+m, i+2m, \ldots, i+(k_m-1)m)$$

where i is the first index of the cycle, and $k_m = n/\gcd(m,n)$. Indeed k_m must be such that $k_m m \equiv 0 \pmod{n}$, which is the case. But also, there cannot be a smaller k' such that $k'm \equiv 0 \pmod{n}$: we need to multiply m by an integer such that the product is $0 \pmod{n}$, and the smallest integer is obtained by multiplying m by the prime factors that are missing to m to obtain n, and only those, which is exactly what k_m does.

Now that we know that each cycle has length k_m (note that the reasoning does not depend on i), and since the union of all cycles must give n, we get that the number $c(g^m)$ of cycles in the decomposition of g^m is

$$\frac{n}{k_m} = \frac{n}{\frac{n}{\gcd(m,n)}} = \gcd(m,n).$$

We thus have an answer to our Problems 1 and 2 of Necklace Enumeration.

Theorem 1.3. *Given n, k two positive integers, the number of (n,k)-necklaces is*

$$\frac{1}{n}\sum_{i=1}^{n} k^{\gcd(n,i)}.$$

When $n = p$ is prime, this simplifies to

$$\frac{1}{p}((p-1)k + k^p).$$

Corollary 1.4. *Let ϕ be the Euler totient function. Then the number of (n,k)-necklaces can be alternatively written as*

$$\frac{1}{n}\sum_{d|n}\phi(d)k^{n/d}.$$

See Exercise 2 for a proof.

1.2 Pólya's Enumeration Theorem

We saw in the previous section that Burnside Lemma is a powerful tool, and in fact it was the key to solve the problem of counting (n,k)-necklaces. It deserves to spend some time to go through its proof, which furthermore contains a useful counting technique.

[**Burnside Lemma**] Let G be a finite group action on a set X. Then

$$|X/G| = \frac{1}{|G|}\sum_{g\in G}|\mathrm{Fix}(g)|,\ \mathrm{Fix}(g) = \{x \in X,\ g * x = x\}.$$

Proof. We have

$$\begin{aligned}
\sum_{g\in G}|\mathrm{Fix}(g)| &= \sum_{g\in G}|\{x\in X,\ g*x=x\}| \\
&= |\{(g,x)\in G\times X,\ g*x=x\}| \\
&= \sum_{x\in X}|\{g\in G,\ g*x=x\}| \\
&= \sum_{x\in X}\frac{|G|}{|\mathrm{Orb}(x)|},\ \text{this is a claim} \\
&= |G|\sum_{x\in X}\frac{1}{|\mathrm{Orb}(x)|} \\
&= |G|\sum_{A\in X/G}\sum_{x\in A}\frac{1}{|A|},\ \text{since } X=\bigcup_{A\in X/G}A \\
&= |G|\sum_{A\in X/G}1 \\
&= |G||X/G|.
\end{aligned}$$

Note that the first lines are a nice combinatorial trick of counting the same set in two different manners. Also the 6th equality uses Lemma 1.1. We are thus left to prove the claim. □

The set involved in the claim to be proven is called *stabilizer* of x:

$$\mathrm{Stab}(x)=\{g\in G,\ g*x=x\}.$$

We prove the claim separately, it is called the Orbit-Stabilizer Theorem.

Proposition 1.5. [Orbit-Stabilizer Theorem] *Let G be a finite group acting on a set X. Then*

$$|\mathrm{Stab}(x)|=\frac{|G|}{|\mathrm{Orb}(x)|}.$$

Proof. Fix $x\in X$, consider $\mathrm{Orb}(x)$, the orbit of x, which contains the elements $g_1*x,\dots,g_n*x$ for $G=\{g_1,\dots,g_n\}$. Look at g_1*x, and gather all the g_i*x such that $g_i*x=g_1*x$, and call A_1 the set that contains all the g_i. Do the same process with g_2*x (assuming g_2 is not already included in A_1), to obtain a set A_2, and iterate until all elements of G are considered. This creates m sets $A_1,\dots,A_m$, which are in fact equivalence classes for the equivalence relation $\sim$ defined on G by $g\sim h \iff g*x=h*x$. We have $m=|\mathrm{Orb}(x)|$, since there is a distinct equivalence class for each distinct $g*x$ in the orbit, and since $A_1,\dots,A_m$ partition G

$$|G|=\sum_{i=1}^{m}|A_i|.$$

Now $|A_i|=|\mathrm{Stab}(x)|$ for all i. Indeed, fix i and $g\in A_i$. Then

$$h\in A_i \iff g*x=h*x \iff x=g^{-1}h*x \iff g^{-1}h\in\mathrm{Stab}(x) \iff h\in g\mathrm{Stab}(x).$$

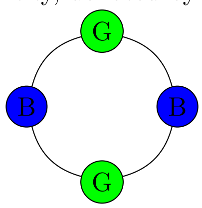
This shows that $|A_i| = |g\text{Stab}(x)| = |\text{Stab}(x)|$, the last equality being a consequence of g being invertible.

Thus

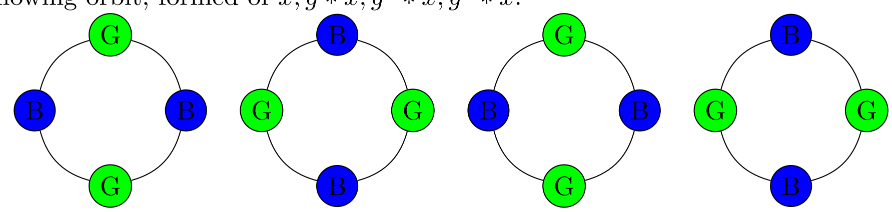
$$|G| = \sum_{i=1}^{m} |A_i| = m|\text{Stab}(x)| = |\text{Orb}(x)||\text{Stab}(x)| \Rightarrow |\text{Orb}(x)| = \frac{|G|}{|\text{Stab}(x)|}.$$

□

Example 1.5. We already saw an illustration of this theorem in Example 1.3. For $n = 4$ and $k = 2$, we considered the 4 rotations (by $\pi/2$, π, $3\pi/2$ and the identity, denoted by $g, g^2, g^3, g^4 = 1$). Then consider the ornament x



on which we apply the 4 rotations, starting from the identity, to get the following orbit, formed of $x, g*x, g^2*x, g^3*x$:



Then $\text{Stab}(x)$ is given by g^2 and $g^4 = 1$, and $|\text{Stab}(x)| = 2 = \frac{|G|}{|\text{Orb}(x)|}$ since the orbit contains only 2 distinct colourings.

The same example can be used to illustrate the proof of the Orbit-Stabilizer Theorem. Let us look again at these 4 ornaments, given by $x, g*x, g^2*x, g^3*x$. Since x and g^2*x give the same colouring, group $1, g^2$ into a set A_1, and since $g*x$ and g^3*x give the same colouring, group g, g^3 into a set A_2. Then $|G| = |A_1| + |A_2|$. We also see that A_1 is actually the stabilizer of x, and that A_2 is $g\text{Stab}(x)$, thus $|A_1| = |A_2| = |\text{Stab}(x)|$, and the number of A_i is the number of distinct colourings in $\text{Orb}(x)$, so $|G| = 2|\text{Stab}(x)| = |\text{Orb}(x)||\text{Stab}(x)|$.

If we look back at the problem of counting necklaces, we used the fact that necklaces can be seen as orbits under the action of a group of rotations, after which we used Burnside's lemma to count the number of orbits. So one can now imagine that the same principle could apply to other counting scenarios, where the group acting is not necessarily the cyclic group. But then, the reasoning on cycles remains the same, so one would have to capture the cycle decomposition of the group elements involved. This is captured by the notion of cycle index.

Definition 1.4. Given a permutation g on n elements, let $c_i(g)$ be the number of cycles of length i in its cycle decomposition. Then the *cycle index* of a permutation group is a polynomial that summarizes the information about the cycle types of all the elements of the group:

$$P_G(X_1, \dots, X_n) = \frac{1}{|G|} \sum_{g \in G} X_1^{c_1(g)} X_2^{c_2(g)} \cdots X_n^{c_n(g)}.$$

Analogies could be that of a weight enumerator for linear codes, or of theta series for lattices. Note that we consider group actions on words of length n, so since the action of g sends a word of length n to another word of length n, g can always be seen as a permutation on n elements (elements in a finite group can always be seen as permutations, remember Cayley Theorem).

Example 1.6. We continue Example 1.4, where $G = C_6$, and we have $n = 6$ beads. Then

$$\begin{array}{rcl} g = & (123456) & c_6(g) = 1 \\ g^2 = & (135)(246) & c_3(g^2) = 2 \\ g^3 = & (14)(25)(36) & c_2(g^3) = 3 \\ g^4 = & (153)(264) & c_3(g^4) = 2 \\ g^5 = & (165432) & c_6(g^5) = 1 \\ g^6 = & (1)(2)(3)(4)(5)(6) & c_1(g^6) = 6 \end{array}$$

and

$$P_G(X_1, \dots, X_6) = \frac{1}{6}(X_6 + X_3^2 + X_2^3 + X_3^2 + X_6 + X_1^6) = \frac{1}{6}(2X_6 + 2X_3^2 + X_2^3 + X_1^6).$$

We may want to pay attention to the information contained in this polynomial. For example, $2X_6$ tells us that we have 2 patterns corresponding to a cycle of length 6 (one is associated to the cycle (123456) and one is associated to the cycle (165432)). Then $2X_3^2$ tells us that we have 2 patterns corresponding to 2 cycles of length 3 (one is associated to the cycle (135)(246), the other to the cycle (153)(264)).

Now let us add the number k of colors. For $2X_6$, we have 2 patterns, each can be of any of the k colours, so this counts $2k$ necklaces. For $2X_3^2$, we also have 2 patterns, but each pattern contains 2 cycles, and each cycle can take one colour, so the number of necklaces is $2k^2$. Iterating the process, we get that the number of $(6, k)$-necklaces is

$$P_G(k, \dots, k) = \frac{1}{6}(2k + 2k^2 + k^3 + k^6),$$

as we already know.

The polynomial $P_G(X_1, \dots, X_n)$ does contain a lot of information, but the difficulty lies in actually finding it.

To state Pólya's Enumeration Theorem in a more general setting than counting coloured necklaces, we will consider D, C two finite sets, G a finite group acting on D, and we will let G act on C^D, the set of functions $f : D \to C$, by

$$(g * f)(d) = f(g^{-1} * d)$$

Figure 1.2: George Pólya (1887-1985). The popularity of the enumeration theorem is in particular due to its applications to chemistry (enumeration of chemical compounds).

(see Exercise 4 for a discussion on this action). The choice of the letters D, C corresponds to a colouring (C) of a domain (D), since a popular application of the theorem is to enumerate coloured objects. Functions $f : D \to C$ by definition of a function must assign a value $f(d) \in C$ for every $d \in D$, so it is the same thing as having $|D|$ slots, and for each slot, assigning a value from C, or said otherwise, we are looking at words of length $|D|$ with alphabet C. For (n, k)-necklaces, $D = \{1, \dots, n\}$ and $C = \{1, \dots, k\}$.

We assign a weight to each element $c \in C$, call it $w(c)$.

Definition 1.5. The weight $W(f)$ of a function $f \in C^D$ is the product

$$W(f) = \prod_{d \in D} w(f(d)).$$

We first notice that functions which belong to the same orbit under the action of G have the same weight, and for that reason, we call these orbits *patterns* (in the necklace setting, an orbit, or pattern, is a necklace). Indeed, suppose f_1, f_2 are in the same orbit under the action of G, that is, there exists $g \in G$ for which $f_2(d) = g * f_1(d)$ for all d:

$$\begin{aligned} W(f_2) &= \prod_{d \in D} w(f_2(d)) = \prod_{d \in D} w(g * f_1(d)) = \prod_{d \in D} w(f_1(g^{-1} * d)) \\ &= \prod_{d \in D} w(f_1(d)) = W(f_1). \end{aligned}$$

The second line equality is saying that when d runs through all values of D, so does $g^{-1} * d$ (if this was not the case, then $g^{-1} * d$ would not go through all possible values of D, and there would exist $d \neq d' \in D$ with $g^{-1} * d = g^{-1} * d'$ which is not possible because g^{-1} is invertible). Thus if F denotes a pattern (an orbit under the action of G on C^D), instead of considering weights $f \in F$, it is enough to consider the weight $W(F)$ of F, which is then $W(f)$ for any choice of f in F.

Example 1.7. Consider again the $(4, 2)$-necklace problem of Example 1.2. We have the set of functions $f : D = \{1, 2, 3, 4\} \to C = \{c_1, c_2\}$, with $|D| = 4$, and $|C| = 2$. Choose weights for $c \in C$, e.g.

$$w(c_1) = R,\ w(c_2) = B,$$

where the weights capture the property of colouring that is of interest in the necklace problem. Pick the function f_1, given by:

$$f_1(1) = c_2,\ f_1(2) = c_2,\ f_1(3) = c_1,\ f_1(4) = c_1,$$

then $w(f_1) = \prod_{i=1}^4 w(f_1(i)) = B^2R^2$. Then pick the function f_2 given by:

$$f_2(1) = c_1,\ f_2(2) = c_1,\ f_2(3) = c_2,\ f_2(4) = c_2,$$

with $w(f_2) = \prod_{i=1}^4 w(f_2(i)) = B^2R^2$. It has the same weights as f_1, and if we take g to be a rotation of 180 degrees clockwise, we get $g * f_1 = (c_1, c_1, c_2, c_2) = f_2$, thus f_1 and f_2 belong to the same pattern. Now pick the function f_3, given by

$$f_3(1) = c_2,\ f_3(2) = c_1,\ f_3(3) = c_2,\ f_3(4) = c_1,$$

with $w(f_3) = \prod_{i=1}^4 w(f_3(i)) = B^2R^2$, which is the same weight as that of f_1, f_2, however f_3 does not belong to the same pattern. To see this, notice that the permutation needed to send f_3 to f_1 cannot be obtained by rotation.

The example illustrates that there is one weight assigned to a pattern because every function in this orbit has the same weight, however several orbits (or patterns) could have the same weight. In a sense, this is saying that the weight is a coarser characterization of functions than patterns.

Theorem 1.6. [Pólya's Enumeration Theorem] *Let D, C be two finite sets, and let G be a finite group acting on C^D. We assign a weight $w(c)$ to each element $c \in C$. The patterns F have induced weights $W(F)$. Then the* pattern inventory *is*

$$\sum_F W(F) = P_G\left(\sum_{c\in C} w(c), \sum_{c\in C} w(c)^2, \sum_{c\in C} w(c)^3, \dots\right),$$

where P_G is the cycle index.

Corollary 1.7. *If all the weights are chosen to be equal to 1, then the number of patterns (or orbits of G on C^D) is given by $P_G(|C|, \dots, |C|)$.*

We first prove the corollary.

Proof. If all weights are equal to 1, then $W(F) = \prod_{d\in D} W(f(d)) = 1$ and $\sum_F W(F) = \sum_F 1$ just counts the number of patterns. Also $\sum_{c\in C} w(c)^i = \sum_{c\in C} 1 = |C|$ for $i \geq 1$. □

We next prove the theorem.

Proof. To evaluate $\sum_F W(F)$, we need to consider all possible weights ω of F, and for each ω, count how many F we have such that $W(F) = \omega$:

$$\sum_F W(F) = \sum_\omega \omega |\{F,\ W(F) = \omega\}|.$$

Counting $|\{F,\ W(F) = \omega\}|$ means counting the number of orbits F under the action of G, restricting to functions of given weight, that is we restrict the action of G on

$$S_\omega = \{f \in C^D, W(f) = \omega\}$$

and $|\{F,\ W(F) = \omega\}| = |S_\omega/G|$. Let us count how many patterns (or orbits) are in S_ω using Burnside Lemma:

$$|S_\omega/G| = \frac{1}{|G|}\sum_{g\in G} |\mathrm{Fix}_\omega(g)|,\ \mathrm{Fix}_\omega(g) = \{f \in C^D,\ W(f) = \omega,\ g * f = f\}.$$

Thus

$$\sum_F W(F) = \sum_\omega \omega \frac{1}{|G|}\sum_{g\in G} |\mathrm{Fix}_\omega(g)| = \frac{1}{|G|}\sum_{g\in G}\sum_\omega \omega |\mathrm{Fix}_\omega(g)|.$$

Since

$$\begin{aligned} \mathrm{Fix}(g) &= \{f \in C^D,\ g * f = f\} \\ &= \sqcup_\omega \{f \in C^D, W(f) = \omega,\ g * f = f\} \\ &= = \sqcup_\omega \mathrm{Fix}_\omega(g), \end{aligned}$$

the sum $\sum_\omega \omega |\mathrm{Fix}_\omega(g)|$ exactly captures all the weights that appear in $\mathrm{Fix}(g)$ for a given g, with their multiplicity.

But this can also be counted as follows: if $x \in \mathrm{Fix}(g)$, then by definition $g * f = f$ and the elements of a cycle of g must be given the same value c by f. A cycle of length i will contribute a factor $\sum_{c\in C} w(c)^i$, capturing that over each of the i terms, $w(c)$ must be the same, thus over the cycle of length i, we have $w(c)^i$, and any choice of $c \in C$ is possible thus $\sum_{c\in C} w(c)^i$. Recalling that $c_i(g)$ denotes the number of cycles of length i, we then have

$$\begin{aligned} \sum_F W(F) &= \frac{1}{|G|}\sum_{g\in G}\left(\sum_{c\in C} w(c)\right)^{c_1(g)} \cdots \left(\sum_{c\in C} w(c)^n\right)^{c_n(g)} \\ &= P_G\left(\sum_{c\in C} w(c), \sum_{c\in C} w(c)^2, \ldots, \sum_{c\in C} w(c)^n\right) \end{aligned}$$

where $n = |D|$. □

Example 1.8. We illustrate the proof using the example of $(4,2)$-necklaces. There are 2^4 possible such necklaces before considering their equivalence up to rotation. We list them by weight:

B^4	RB^3	R^2B^2	R^3B	R^4
$BBBB$	$RBBB$	$RRBB$	$RRRB$	$RRRR$
	$BRBB$	$RBRB$	$RRBR$	
	$BBRB$	$BRRB$	$RBRR$	
	$BBBR$	$RBBR$	$BRRR$	
		$BRBR$		
		$BBRR$		

What the proof of Pólya's Enumeration Theorem does, is to look at each column, they correspond to S_ω, for ω ranging from B^4 to R^4. Now looking at the action of $G = C_4$ on each column, we see that $|S_\omega/G| = 1$ except for $\omega = R^2B^2$, in which case $|S_\omega/G| = 2$. Thus we have 5 different weights, but 6 different patterns, since two patterns have the same weight. This illustrates:

$$\sum_F W(F) = \sum_\omega \omega|S_\omega/G| = B^4 + RB^3 + 2R^2B^2 + R^3B + R^4.$$

This can be computed differently, by looking at every weight in $\mathrm{Fix}(g)$. For a given g, if $x \in \mathrm{Fix}(g)$, it must be that the weights are the same for every elements of a cycle of g:

- for a cycle of length 1, the possible weights are $R + B$,
- for a cycle of length 2, the possible weights are $R^2 + B^2$,
- for a cycle of length 4, the possible weights are $R^4 + B^4$.

Let us thus look at the possible group elements, and their cycle information ($c_i(g)$ is the number of cycles of length i in the cycle decomposition of g):

$$\begin{array}{ccc}
1 & c_1(1) = 4 & (R+B)^4 \\
g & c_4(g) = 1 & R^4 + B^4 \\
g^2 & c_2(g) = 2 & (R^2+B^2)^2 = R^4 + 2R^2B^2 + B^4 \\
g^3 & c_4(g) = 1 & R^4 + B^4
\end{array}$$

- For 1, we have 4 cycles of length 1, thus $(R+B)^4$. Also, $(R+B)^4 = (R^2+2RB+B^2)^2 = R^4 + 4R^3B + 6R^2B^2 + 4RB^3 + B^4$ gives all possible weights of length 4 involving 2 colours.
- For g and g^3, we have 1 cycle of length 4, thus $(R^4 + B^4)$. This means that g and g^3 fix only $RRRR$ and $BBBB$.
- For g^2, we have two cycles of length 2, thus $(R^2+B^2)^2$: g^2 does fix $RRRR$ and $BBBB$ but also 2 patterns of weight R^2B^2.

If we sum over all $g \in G$ their corresponding weight, we get:

$$(R^4 + 4R^3B + 6R^2B^2 + 4RB^3 + B^4) + 2(R^4 + B^4) + (R^4 + 2R^2B^2 + B^4)$$

which simplifies to

$$4R^4 + 4R^3B + 8R^2B^2 + 4RB^3 + 4B^4$$

as expected.

Let us next just apply the theorem. The cycle index is

$$P_G(X_1, \dots, X_4) = \frac{1}{|G|}\sum_{g \in G} X_1^{c_1(g)} \cdots X_1^{c_4(g)} = \frac{1}{|G|}(X_1^4 + 2X_4 + X_2^2).$$

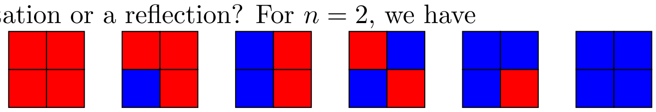
We have two colours R and B, then $\sum_c w(c)^i = R^i + B^i$, and we just need to evaluate

$$\begin{aligned} & \frac{1}{4}P_G(B+R, B^2+R^2, B^3+R^3, B^4+R^4) \\ = \quad & \frac{1}{|G|}((B+R)^4 + 2(B^4+R^4) + (B^2+R^2)^2) \\ = \quad & R^4 + R^3B + 2R^2B^2 + RB^3 + B^4. \end{aligned}$$

This tells us for example that there are two possible necklaces with two beads of each colour.

Let us see if we can use Pólya's Theorem for counting something else than (n,k)-necklaces.

Example 1.9. Consider an $n \times n$ chessboard, $n \geq 2$, where every square is either colored by blue (B) or red (R). How many different colourings are there, if different means that one cannot obtain a colouring from another by either a rotation or a reflection? For $n = 2$, we have



We first to observe that:

- There are 4 rotations, r the rotation by 90 degrees clockwise, r^2 the rotation by 180 clockwise, r^3 the rotation by 270 degrees clockwise, and r^4 the identity.

- There are 4 reflections: one with respect to the vertical axis, one with respect to the horizontal axis, and two with respect to each of the two diagonals. Let m be the reflection with respect to the horizontal axis. Note that rm, that is applying first m and then r, gives a reflection with respect to the left diagonal, that r^3m gives a reflection with respect to the right diagonal, and that r^2m gives a reflection with respect to the vertical axis.

- The 8 reflections and rotations are thus given by

$$\{1, r, r^2, r^3, m, rm, r^2m, r^3m\}.$$

In order to apply Pólya's enumeration Theorem, we need the action of a group G, for which we need to show that $G = \{1, r, r^2, r^3, m, rm, r^2m, r^3m\}$ is a group (see Exercise 5). We then need to compute $\mathrm{Fix}(g)$ for each $g \in G$, for which we write the cycle decomposition. Note that we label the 4 squares of the 2×2 chessboard clockwise:

1	2
4	3

so that a permutation by r is of the form (1234). We get

g		$\|\mathrm{Fix}(g)\|$
1	(1)(2)(3)(4)	2^4
r	(1234)	2
r^2	(13)(24)	2^2
r^3	(1432)	2
m	(14)(23)	2^2
r^2m	(12)(34)	2^2
rm	(24)(1)(3)	2^3
r^3m	(13)(2)(4)	2^3

Using Burnside lemma, we get:

$$\frac{1}{8}(2^4 + 2 + 2^2 + 2 + 2^2 + 2^2 + 2^3 + 2^3) = \frac{48}{6} = 6.$$

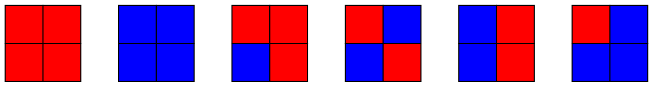
We can compute the cycle index:

$$P_G(X_1, X_2, X_3, X_4) = \frac{1}{8}(X_1^4 + 2X_1^2X_2 + 3X_2^2 + 2X_4)$$

which evaluated in $X_1 = X_2 = X_3 = X_4 = 2$ gives 6 as expected. Now using Pólya's enumeration Theorem:

$$\begin{aligned}
& P_G((R+B), (R^2+B^2), (R^3+B^3), (R^4+B^4)) \\
= & \frac{1}{8}((R+B)^4 + 2(R+B)^2(R^2+B^2) + 3(R^2+B^2)^2 + 2(R^4+B^4)) \\
= & \frac{1}{8}(R^4 + 4R^3B + 6R^2B^2 + 4RB^3 + B^4) + \\
& \frac{1}{8}(2R^4 + 2R^2B^2 + 4R^3B + 4RB^3 + 2B^2R^2 + 2B^4) + \\
& \frac{1}{8}(3R^4 + 3B^4 + 6R^2B^2 + 2R^4 + 2B^4) \\
= & R^4 + B^4 + R^3B + 2R^2B^2 + RB^3.
\end{aligned}$$

This enumerates the 6 colourings of the chessboard:



The case $n = 2$ could be done by hand, this is much harder in general (see Exercise 6).

1.3 Exercises

Exercise 1. Compute the number of $(6, 3)$-necklaces, that is the number of necklaces with 6 beads and 3 colours.

Exercise 2. Prove that

$$\frac{1}{n}\sum_{d|n}\phi(d)k^{n/d}$$

counts the number of (n, k)-necklaces, where ϕ is Euler totient function.

Exercise 3. Compute the cycle index $P_{C_n}(X_1, \ldots, X_n)$.

Exercise 4. Let D, C be two finite sets, let G be a group acting on D, and let C^D be the set of functions $f : D \to C$. Show that

$$(g * f)(d) = f(g^{-1} * d)$$

is indeed a group action on C^D.

Exercise 5. Show that

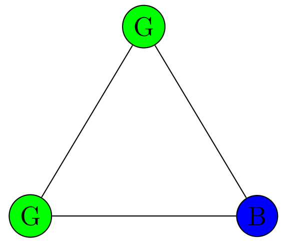
$$\{1, r, r^2, r^3, m, rm, r^2m, r^3m\}$$

form a group with respect to composition of maps, where r is a rotation by 90 degrees (clockwise) and m is a reflection through the horizontal axis.

Exercise 6. Consider an $n \times n$ chessboard, $n \geq 2$, where every square is either colored by blue (B) or red (R). How many different colorings are there, if different means that one cannot obtain a coloring from another by either a rotation (by either 90, 180 or 270 degrees) or a reflection (along the vertical and horizontal axes, and the 2 diagonals)?

Exercise 7. Consider $(4, 3)$-necklaces, that is the number of necklaces with 4 beads and 3 colours, say blue (B), green (G) and red (R) . Use Pólya's Enumeration Theorem to list the different necklaces involving at least two blue beads.

Exercise 8. Consider an equilateral triangle whose vertices are coloured, they can be either blue or green. Here is an example of colouring:



Two colourings of the vertices are considered equivalent if one can be obtained from another via a rotation or reflection of the triangle.

1. List all the rotation(s) and reflection(s) of the equilateral triangle and argue they form a group.
2. Compute the cycle index polynomial for the group of rotations and reflections of the equilateral triangle.
3. Use Polya's Enumeration Theorem to list the different colourings using two colours of the equilateral triangle.

We know that a (n, k)-necklace is an equivalence class of words of length n over an alphabet size k, under rotation. Consider now an (n, k)-bracelet, that is an equivalence class of words of length n over an alphabet of size k, under both rotation (as necklaces), but also under reversal, which means for example that the bracelet $ABCD$ is equivalent to the bracelet $DCBA$: $ABCD \equiv DCBA$.

Exercise 9. We know that a (n, k)-necklace is an equivalence class of words of length n over an alphabet size k, under rotation. Consider now an (n, k)-bracelet, that is an equivalence class of words of length n over an alphabet of size k, under both rotation (as necklaces), but also under reversal, which means for example that the bracelet $ABCD$ is equivalent to the bracelet $DCBA$: $ABCD \equiv DCBA$.

1. For (4,2)-necklaces, they are orbits under the action of the group of rotations by $(2\pi/4)k, k = 0, 1, 2, 3$. For (4,2)-bracelets, they are also orbits under the action of some group G. What is this group G? List its elements.
2. Compute the cycle index polynomial for the group G.
3. Use Polya's Enumeration Theorem to list the different (4,2)-bracelets.