Chapter 10

Functions

> *"One of the most important concepts in all of mathematics is that of function." (T.P. Dick and C.M. Patton)*

Functions...finally a topic that most of you must be familiar with. However here, we will not study derivatives or integrals, but rather the notions of one-to-one and onto (or injective and surjective), how to compose functions, and when they are invertible.

Let us start with a formal definition.

Definition 63. Let X and Y be sets. A function f from X to Y is a rule that assigns every element x of X to a unique y in Y. We write $f : X \to Y$ and $f(x) = y$. Formally, using predicate logic:

$$(\forall x \in X,\ \exists y \in Y,\ y = f(x)) \wedge (\forall x_1, x_2 \in X,\ f(x_1) \neq f(x_2) \to x_1 \neq x_2).$$

Then X is called the domain of f, and Y is called the codomain of f. The element y is the image of x under f, while x is the preimage of y under f. Finally, we call range the subset of Y with preimages.

Example 96. Consider the assignment rule $f : X = \{a, b, c\} \to Y = \{1, 2, 3, 4\}$ which is defined by: $f = \{(a, 2), (b, 4), (c, 2)\}$. We first check that this is a function. For every element in X, we do have an assignment: $f(a) = 2,\ f(b) = 4,\ f(c) = 2$. Then the condition that whenever $f(x_1) \neq f(x_2)$ it must be that $x_1 \neq x_2$ is also satisfied. The the domain of f is X, the codomain of f is Y. The preimage of 2 is $\{a, c\}$ because $f(a) = f(c) = 2$. For the range, we look at Y, and among $1, 2, 3, 4$, only 2 and 4 have a preimage, therefore the range is $\{2, 4\}$.

Function

Let X and Y be sets. A **function** f from X to Y is a rule that assigns every element x of X *to a unique y in Y*. We write $\boldsymbol{f: X \to Y}$ and $\boldsymbol{f(x) = y}$

$$(\forall x \in X\ \exists y \in Y, y = f(x)) \wedge (\forall x_1, x_2 \in X, f(x_1) \neq f(x_2) \rightarrow x_1 \neq x_2)$$

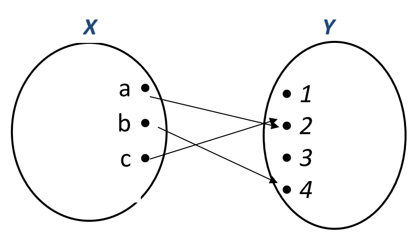
X = domain, Y = codomain
y = image of x under f,
x = preimage of y under f
range = subset of Y with preimages

Example 1

$$(\forall x \in X\ \exists y \in Y, y = f(x)) \wedge (\forall x_1, x_2 \in X, f(x_1) \neq f(x_2) \rightarrow x_1 \neq x_2)$$

Domain X={a,b,c},
Co-domainY={1,2,3,4}
f={(a,2),(b,4),(c,2)},
preimage of 2 is{a,c}
Range={2,4}

Arrow Diagram of f:



Example 97. The rule f that assigns the square of an integer to this integer is a function. Indeed, every integer has an image: its square. Also whenever two squares are different, it must be that their square roots were different. We write

$$f : \mathbb{Z} \to \mathbb{Z},\ f(x) = x^2.$$

Its domain is $\mathbb{Z}$, its codomain is $\mathbb{Z}$ as well, but its range is $\{0, 1, 4, 9, 16, \ldots\}$, that is the set of squares in $\mathbb{Z}$.

Definition 64. Let f be a function from X to Y, X, Y two sets, and consider the subset $S \subset X$. The image of the subset S is the subset of Y that consists of the images of the elements of S: $f(S) = \{f(s),\ s \in S\}$

We next move to our first important definition, that of one-to-one.

Definition 65. A function f is one-to-one or injective if and only if $f(x) = f(y)$ implies $x = y$ for all x, y in the domain X of f. Formally:

$$\forall x, y \in X(f(x) = f(y) \to x = y).$$

In words, this says that all elements in the domain of f have different images.

Example 98. Consider the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 4x - 1$. We want to know whether each element of $\mathbb{R}$ has a different image. Yes, this is the case, why? well, visually, this function is a line, so one may "see" that two distinct elements have distinct images, but let us try a proof of this. We have to show that $f(x) = f(y)$ implies $x = y$. Ok, let us take $f(x) = f(y)$, that is two images that are the same. Then $f(x) = 4x - 1$, $f(y) = 4y - 1$, and thus we must have $4x - 1 = 4y - 1$. But then $4x = 4y$ and it must be that $x = y$, as we wanted. Therefore f is injective.

Example 99. Consider the function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$. Do we also have that two distinct reals have distinct images? Well no... because 1 and -1 are both sent to 1...so this function is not injective! If $g(x) = g(y) = 1$, we cannot conclude that $x = y$, in fact this is wrong, it could be that $x = -y$.

The other definition that always comes in pair with that of one-to-one/injective is that of onto.

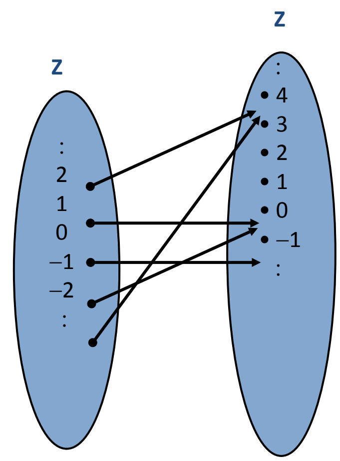
Example 2

Let *f* be the function from **Z** to **Z** that assigns the square of an integer to this integer.

Then, *f*: **Z** → **Z**, $f(x) = x^2$

Domain and co-domain of *f*: **Z**

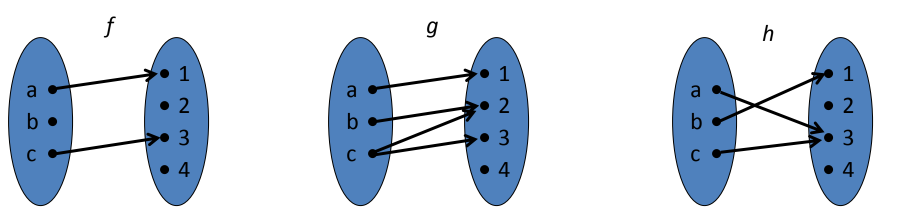
Range(*f*) = {0, 1, 4, 9, 16, 25, ….}



Functions Vs Non-functions

$(\forall x \in X\ \exists y \in Y, y = f(x)) \wedge (\forall x_1, x_2 \in X, f(x_1) \neq f(x_2) \rightarrow x_1 \neq x_2)$

X = {a,b,c} to *Y* = {1,2,3,4}



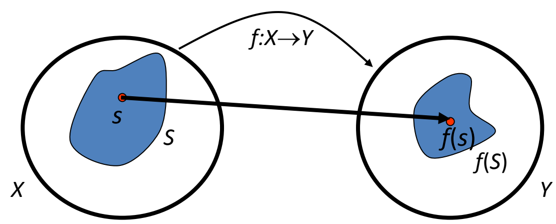
No,
b has no image

No,
c has two images

Yes,
each element of *X* has exactly one image

Image of a Set

Let ***f*** be a functions from *X* to *Y* and $S \subseteq X$. The **image of S** is the subset of *Y* that consists of the images of the elements of *S*: $\boldsymbol{f(S) = \{f(s) \mid s \in S\}}$



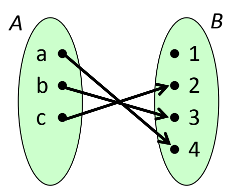
One-To-One Function

A function *f* is **one-to-one** (or **injective**), if and only if $f(x) = f(y)$ implies $x = y$ for all *x* and *y* in the domain of *f*.

In words:
"All elements in the domain of f have different images"

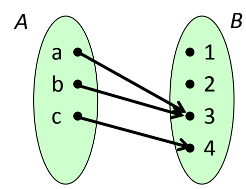
Mathematical Description:
$f{:}A \to B$ is **one-to-one** $\Leftrightarrow \forall x_1, x_2 \in A \; (f(x_1)=f(x_2) \rightarrow x_1 = x_2)$
or
$f{:}A \to B$ is **one-to-one** $\Leftrightarrow \forall x_1, x_2 \in A \; (x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2))$

Example: One-to-One (Injective)



one-to-one

(all elements in *A* have a different image)



not one-to-one

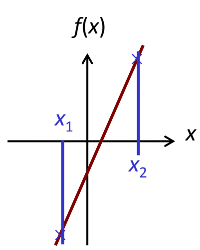
(*a* and *b* have the same image)

Example: One-To-One (Injective)

f: **R** $\to$ **R**, $f(x) = 4x-1$

g: **R** $\to$ **R**, $g(x) = x^2$

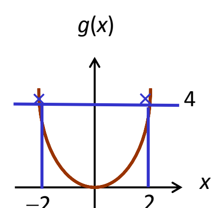
("Does each element in **R** have a different image ?")



Yes !

To show: $\forall x_1, x_2 \in$ **R** $(f(x_1)=f(x_2) \Rightarrow x_1 = x_2)$

Take some $x_1, x_2 \in$ R with $f(x_1) = f(x_2)$.
Then $4x_1-1 = 4x_2-1 \Rightarrow 4x_1 = 4x_2 \Rightarrow x_1=x_2$



No !

Take $x_1 = 2$ and $x_2 = -2$.

Then $g(x_1) = 2^2 = 4 = g(x_2)$
and $x_1 \neq x_2$

Definition 66. A function f is onto or surjective if and only if for every element $y \in Y$, there is an element $x \in X$ with $f(x) = y$:

$$\forall y \in Y,\ \exists x \in X,\ f(x) = y.$$

In words, each element in the co-domain of f has a pre-image.

Example 100. Consider again the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 4x - 1$. We want to know whether each element of $\mathbb{R}$ has a preimage. Yes, it has, let us see why: we want to show that there exists x such that $f(x) = 4x - 1 = y$. Given y, we have the relation $x = (y + 1)/4$ thus this x is indeed sent to y by f.

Example 101. Consider again the function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$. Does each element in $\mathbb{R}$ have a preimage? well, again no... Because $\mathbb{R}$ contains all the negative real numbers, and it is not possible to square a real number and get something negative... Formally, if $y = -1$, there is no $x \in \mathbb{R}$ such that $g(x) = x^2 = -1$.

We next combine the definitions of one-to-one and onto, to get:

Definition 67. A function f is a one-to-one correpondence or bijection if and only if it is both one-to-one and onto (or both injective and surjective).

An important example of bijection is the identity function.

Definition 68. The identity function i_A on the set A is defined by:

$$i_A : A \to A,\ i_A(x) = x.$$

Example 102. Consider the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 4x - 1$, which we have just studied in two examples. We know it is both injective (see Example 98) and surjective (see Example 100), therefore it is a bijection.

Bijections have a special feature: they are invertible, formally:

Definition 69. Let $f : A \to B$ be a bijection. Then the inverse function of f, $f^{-1} : B \to B$ is defined elementwise by: $f^{-1}(b)$ is the unique element $a \in A$ such that $f(a) = b$. We say that f is invertible.

Note the importance of the hypothesis: f must be a bijection, otherwise the inverse function is not well defined. For example, if f is not one-to-one, then $f^{-1}(b)$ will have more than one value, and thus is not properly defined.

Note that given a bijection $f : A \to B$ and its inverse $f^{-1} : B \to A$, we can write formally the above definition as:

$$\forall b \in B,\ \forall a \in A(f^{-1}(b) = a \iff b = f(a)).$$

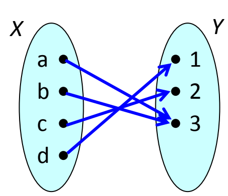
Onto Functions

A function f from X to Y is **onto** (or **surjective**), if and only if for every element $y \in Y$ there is an element $x \in X$ with $f(x)=y$.

In words:
"Each element in the co-domain of f has a pre-image"

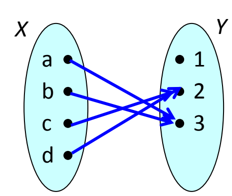
Mathematical Description:
$f:X \to Y$ is **onto** $\Leftrightarrow \forall y \, \exists x, \; f(x) = y$

Example: Onto (Surjective)



onto

(all elements in Y have a pre-image)



not onto

(1 has no pre-image)

Example: Onto (Surjective)

g:**R**$\to$**R**, $g(x)=x^2$

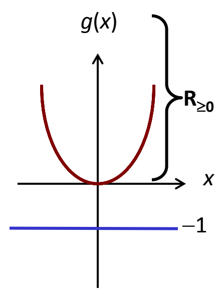
(“Does each element in **R** have a pre-image ?”)

No !

To show: $\exists y \in$ **R** such that $\forall x \in$ **R** $g(x) \neq y$

Take $y = -1$

Then any $x\in$**R** holds $g(x) = x^2 \neq -1 = y$



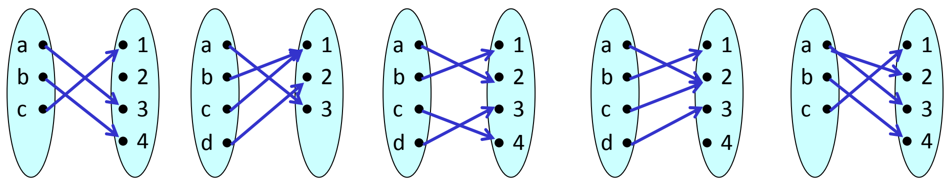
But g:**R**$\to$**R**$_{\geq 0}$, $g(x)=x^2$,(where **R**$_{\geq 0}$ denotes the set of non-negative real numbers) is onto !

One-to-one Correspondence

A function f is a **one-to-one correspondence** (or **bijection**), if and only if it is both one-to-one and onto

In words:
“No element in the co-domain of f has two (or more) pre-images” (*one-to-one*) and
“Each element in the co-domain of f has a pre-image” (*onto*)

Example: Bijection

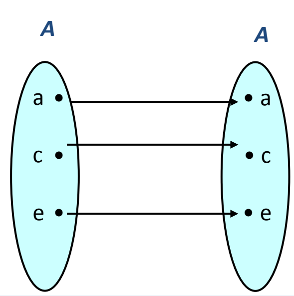


No	No	Yes	No	No
(not onto, 2 has no pre-image)	(not one-to-one, 1 has two pre-images)	(each element has exactly one pre-image)	(neither one-to-one nor onto)	(not a function, *a* has two images)

Identity Function

The **identity function** on a set *A* is defined as: $i_A: A \to A,\ i_A(x) = x$.

Example. Any identity function is a bijection. e.g. for *A* = {a, c, e}:



Example 103. Let us look again at our two previous examples, namely, $f(x) = 4x - 1$ and $g(x) = x^2$. Then $g(x)$, for $g : \mathbb{R} \to \mathbb{R}$ is not a bijection, so it cannot have an inverse. Now $f(x)$ is a bijection, so we can compute its inverse. Suppose that $y = f(x)$, then

$$y = 4x - 1 \iff y + 1 = 4x \iff x = \frac{y+1}{4},$$

and $f^{-1}(y) = \frac{y+1}{4}$.

We saw that for the notion of inverse f^{-1} to be defined, we need f to be a bijection. The next result shows that f^{-1} is a bijection as well.

Proposition 1. *If $f : X \to Y$ is a one-to-one correspondence, then $f^{-1} : Y \to X$ is a one-to-one correspondence.*

Proof. To prove this, we just apply the definition of bijection, namely, we need to show that f^{-1} is an injection, and a surjection. Let us start with injection.

- f^{-1} is an injection: we have to prove that if $f^{-1}(y_1) = f^{-1}(y_2)$, then $y_1 = y_2$. All right, then $f^{-1}(y_1) = f^{-1}(y_2) = x$ for some x in X. But $f^{-1}(y_1) = x$ means that $y_1 = f(x)$, and $f^{-1}(y_2) = x$ means that $y_2 = f(x)$, by definition of the inverse of function. But this shows that $y_1 = y_2$, as needed.

- f^{-1} is an surjection: by definition, we need to prove that any $x \in X$ has a preimage, that is, there exists y such that $f^{-1}(y) = x$. Because f is a bijection, there is some y such that $y = f(x)$, therefore $x = f^{-1}(y)$ as needed.

□

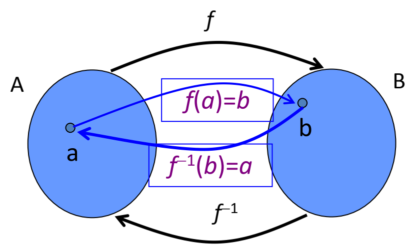
Suppose that you have two functions f and g. It may be possible to compose them to obtain a third function, here is how:

Definition 70. Let $f : A \to B$ be a function, and $g : B \to C$ be a function. Then the composition of f and g is a new function denoted by $g \circ f$, and defined by: $g \circ f : A \to C$, $(g \circ f)(a) = g(f(a))$.

Note that the codomain of f is B, which is the domain of g. Under this condition, the composition $g \circ f$ consists of applying first f, and then apply g on the result. Therefore, $g \circ f \neq f \circ g$ in general!

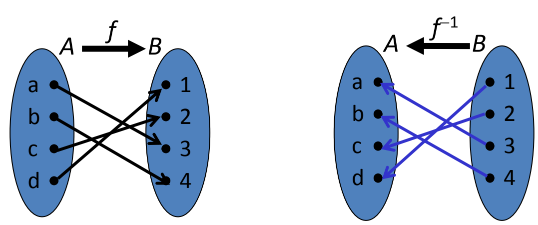
Inverse Function

Let f:$A \rightarrow B$ be a *one-to-one correspondence (bijection).*
Then the inverse function of f, f^{-1}:$B \rightarrow A$, is defined by:
$f^{-1}(b)$ = that unique element $a \in A$ such that $f(a)=b$.
We say that f is invertible.



Example 1

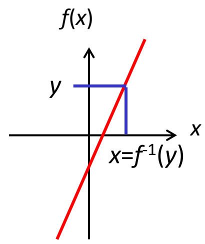
Find the inverse function of the following function:



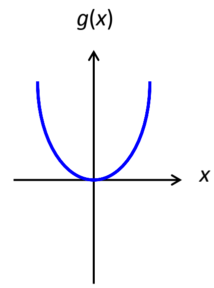
Let f:$A \rightarrow B$ be a one-to-one correspondence and f^{-1}:$B \rightarrow A$ its inverse. Then $\forall b \in B \ \forall a \in A \ (f^{-1}(b)=a \Leftrightarrow b=f(a))$

Example 2

What is the inverse of f:**R**→**R**, $f(x)=4x-1$?



What is the inverse of g:**R**→**R**, $g(x)=x^2$?



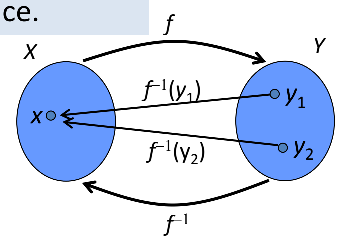
Let $y \in$ **R.** Calculate x with $f(x)=y$:

$y = 4x-1 \Leftrightarrow (y+1)/4 = x$

Hence, $f^{-1}(y) = (y+1)/4$

One-to-one Correspondence

Theorem 1: If $f:X \rightarrow Y$ is a one-to-one correspondence, then $f^{-1}:Y \rightarrow X$ is a one-to-one correspondence.

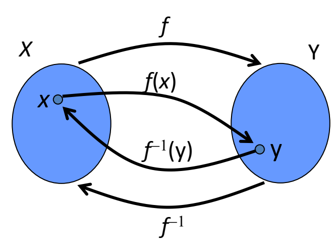


Proof:

(a) f^{-1} is one-to-one:

Take $y_1, y_2 \in Y$ such that $f^{-1}(y_1) = f^{-1}(y_2)$=x.

Then $f(x) = y_1$ and $f(x) = y_2$, thus $y_1 = y_2$.



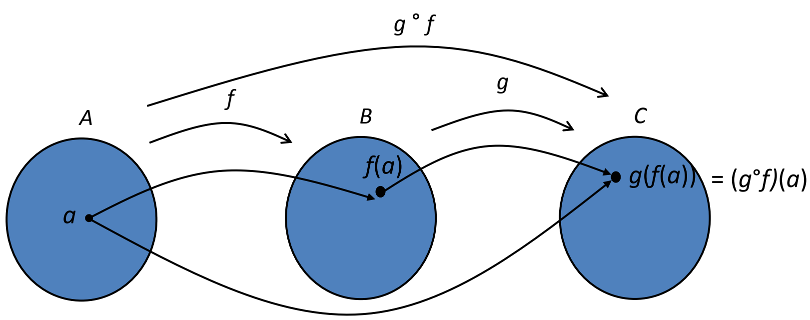
(b) To show f^{-1} is onto:

Take some $x \in X$, and let $y = f(x)$.

Then $f^{-1}(y) = x$.

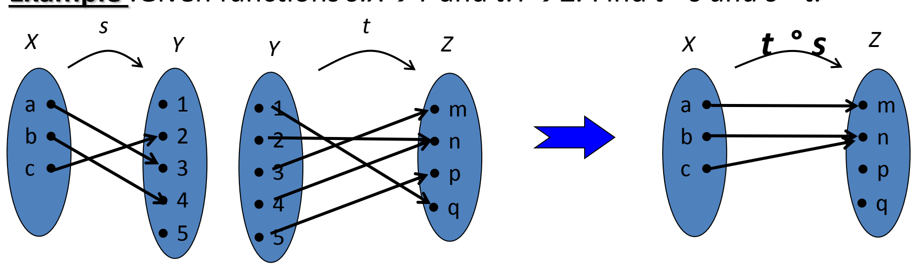
Composition of Functions

Let f:$A \rightarrow B$ and g:$B \rightarrow C$ be functions. The **composition** of the functions f and g, denoted as $g \circ f$, is defined by:
$g \circ f$: $A \rightarrow C$, $(g \circ f)(a) = g(f(a))$



Examples

Example :Given functions s:$X \rightarrow Y$ and t:$Y \rightarrow Z$. Find $t \circ s$ and $s \circ t$.



Example f:**Z**$\rightarrow$**Z**, $f(n)=2n+3$, g:**Z**$\rightarrow$**Z**, $g(n)=3n+2$. What is $g \circ f$ and $f \circ g$?

$(f \circ g)(n) = f(g(n)) = f(3n + 2) = 2(3n+2) + 3 = 6n + 7$

$(g \circ f)(n) = g(f(n)) = g(2n + 3) = 3(2n+3) + 2 = 6n + 11$

$f \circ g \neq g \circ f$ (no **commutativity** for the composition of functions !)

Example 104. Consider $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = 2n+3$, $g(n) = 3n + 2$. We have

$$(f \circ g)(n) = f(g(n)) = f(3n + 2) = 2(3n + 2) + 3 = 6n + 7,$$

while

$$(g \circ f)(n) = g(f(n)) = g(2n + 3) = 3(2n + 3) + 2 = 6n + 11.$$

Suppose now that you compose two functions f, g, and both of them turn out to be injective. The next result tells us that the combination will be as well!

Proposition 2. *Let $f : X \to Y$ and $g : Y \to Z$ be two injective functions. Then $g \circ f$ is also injective.*

Proof. What we need to do is check the injectivity of a function, so we do this as usual: we check that $g \circ f(x_1) = g \circ f(x_2)$ implies $x_1 = x_2$. Typically, to be able to prove this, you will have to keep in mind assumptions, namely that both f and g are injective. So let us start. We have $g \circ f(x_1) = g \circ f(x_2)$ or equivalently $g(f(x_1)) = g(f(x_2))$. But we know that g is injective, so this implies $f(x_1) = f(x_2)$. Next we use that f is injective, thus $x_1 = x_2$, as needed! □

Let us ask the same question with surjectivity, namely whether the composition of two surjective functions gives a function which is surjective too. Here is the answer:

Proposition 3. *Let $f : X \to Y$ and $g : Y \to Z$ be two surjective functions. Then $g \circ f$ is also surjective.*

Proof. The codomain of $g \circ f$ is Z, therefore we need to show that every $z \in Z$ has a preimage x, namely that there always exists an x such that $g \circ f(x) = z$. Again, we keep in mind that f and g are both surjective. Since g is surjective, we know there exists $y \in Y$ such that $g(y) = z$. Now again, since f is surjective, we know there exists $x \in X$ such that $f(x) = y$. Therefore there exist x, y such that $z = g(y) = g(f(x))$ as needed. □

One-to-one Propagation

Theorem 2: Let $f:X \rightarrow Y$ and $g:Y \rightarrow Z$ be both one-to-one functions. Then $g \circ f$ is also one-to-one.

Proof: to show: $\forall x_1, x_2 \in X\ ((g \circ f)(x_1) = (g \circ f)(x_2) \Rightarrow x_1 = x_2)$

Suppose $x_1, x_2 \in X$ with $(g \circ f)(x_1) = (g \circ f)(x_2)$.

Then $g(f(x_1))=g(f(x_2))$.

Since g is one-to-one, it follows $f(x_1) = f(x_2)$.

Since f is one-to-one, it follows $x_1 = x_2$.

Onto Propagation

Theorem 3: Let $f:X \rightarrow Y$ and $g:Y \rightarrow Z$ be both onto functions. Then $g \circ f$ is also onto.

Proof: to show: $\forall z \in Z\ \exists x \in X$ such that $(g \circ f)(x) = z$

Let $z \in Z$.

Since g is onto $\exists y \in Y$ with $g(y)=z$.

Since f is onto $\exists x \in X$ with $f(x)=y$.

Hence, with $(g \circ f)(x) = g(f(x)) = g(y) = z$.

We conclude this chapter on functions, by discussing the pigeonhole principle.

Definition 71. The pigeonhole principle states the following: if you have k pigeonholes, and n pigeons, but the number n of pigeons is more than the number k of pigeonholes, then at least one pigeonhole contains at least two pigeons.

Here is a simple illustration: if you have 4 pigeons and 3 pigeonholes:

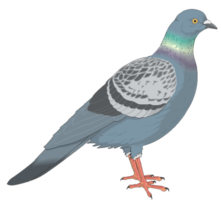
1. Put the first pigeon in the first pigeonhole, if the second pigeon is also here, then we are done, we have at least one pigeonhole with at least 2 pigeons.

2. If the second pigeon went into the second pigeonhole, repeat the argument: if the third pigeon is also here, then we are done, we have at least one pigeonhole with at least 2 pigeons.

3. If the third pigeon went into the third pigeonhole, then at this time, you have 3 pigeonholes, each containing one pigeon, therefore no matter where the fourth pigeon will go, we have at least one pigeonhole with at least 2 pigeons!

This principle is attributed to the mathematician Dirichlet, and is actually very powerful. It is a consequence of the fact that a function from a finite set (the number of pigeons) to a smaller finite set (the number of pigeonholes) cannot be one-to-one, meaning that there must be at least two elements (two pigeons) in the domain, that have the same image (the same pigeonhole) in the co-domain!

Pigeonhole Principle



k pigeonholes, n pigeons, n>k
at least one pigeonhole
contains at least two pigeons

Peter Gustav
Lejeune Dirichlet
(1805-1859)

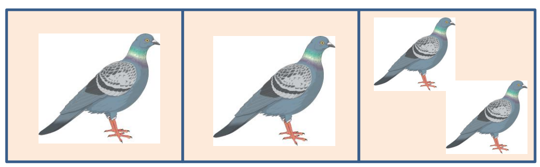
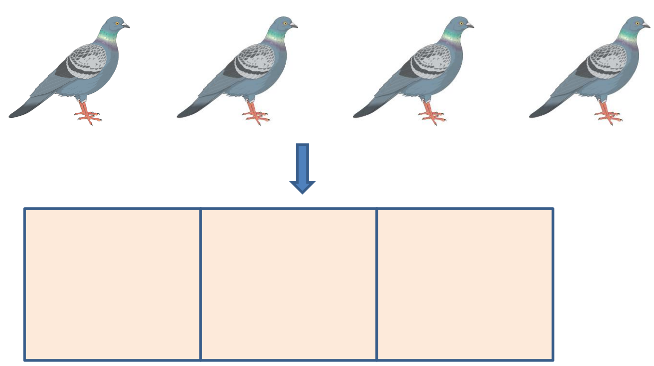


image belongs to the artist, Dirichlet portrait comes from wiki

Pigeonhole Principle

A function from one finite set to a smaller finite set cannot be one-to-one: there must be at least two elements in the domain that have the same image in the co-domain.



Examples

Consider Thorin and his 12 dwarf companions.

- At least two of the dwarves were born on the same day of the week.

- They go to sleep at the Prancing Pony Inn. Thorin gets a room of his own, but the others got to share 4 rooms. Then there are at least 3 dwarves sleeping in at least one of them.

This image belongs to the Hobbit movie

Example 105. Consider Thorin and his 12 dwarf companions.

- At least two of the dwarves were born on the same day of the week: this is consequence of the pigeonhole principle. You have 7 days of the week, and more than 7 dwarves, therefore 7 of them at most could be born each on one day of the week, but the 8th one will necessarily have to share the same day of the week as birthday.

- They sleep at the Prancing Pony Inn, Thorin gets a room of his own (of course, he is the chief!) but the 12 others got to share 4 rooms. Then at least 3 dwarves sleep in at least one room. This is again a consequence of the pigeonhole principle. Imagine room 1, room 2, room 3 and room 4, and 12 dwarves have to fit. The first 4 dwarves could choose room 1, 2, 3, and 4, and be alone in each room. But then the next 4 dwarves will add up, and we will have 2 dwarves in each room. Then no matter how, at least 3 dwarves will end up in one room!

Exercises for Chapter 10

Exercise 85. Consider the set $A = \{a, b, c\}$ with power set $P(A)$ and $\cap$: $P(A) \times P(A) \to P(A)$. What is its domain? its co-domain? its range? What is the cardinality of the pre-image of $\{a\}$?

Exercise 86. Show that $\sin : \mathbb{R} \to \mathbb{R}$ is not one-to-one.

Exercise 87. Show that $\sin : \mathbb{R} \to \mathbb{R}$ is not onto, but $\sin : \mathbb{R} \to [-1, 1]$ is.

Exercise 88. Is $h : \mathbb{Z} \to \mathbb{Z}$, $h(n) = 4n - 1$, onto (surjective)?

Exercise 89. Is $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$, a bijection (one-to-one correspondence)?

Exercise 90. Consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ and $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x + 5$. What is $g \circ f$? What is $f \circ g$?

Exercise 91. Consider $f : \mathbb{Z} \to \mathbb{Z}$, $f(n) = n + 1$ and $g : \mathbb{Z} \to \mathbb{Z}$, $g(n) = n^2$. What is $g \circ f$? What is $f \circ g$?

Exercise 92. Given two functions $f : X \to Y$, $g : Y \to Z$. If $g \circ f : X \to Z$ is one-to-one, must both f and g be one-to-one? Prove or give a counter-example.

Exercise 93. Show that if $f : X \to Y$ is invertible with inverse function $f^{-1} : Y \to X$, then $f^{-1} \circ f = i_X$ and $f \circ f^{-1} = i_Y$.

Exercise 94. If you pick five cards from a deck of 52 cards, prove that at least two will be of the same suit.

Exercise 95. If you have 10 black socks and 10 white socks, and you are picking socks randomly, you will only need to pick three to find a matching pair.