Chapter 7

Complex Numbers

> *"I tell you, with complex numbers you can do anything."* *(J. Derbyshire, Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics)*

So far, the largest set of numbers we have seen is that of real numbers. This will change in this chapter, with the introduction of complex numbers. They were introduced around 1545 by the mathematician Gerolamo Cardano, in order to obtain closed form expressions for roots of cubic polynomial equations, which need square roots of negative numbers, which do not exist (meaning do no exist if only real numbers are considered).

Recall that there is no real number z such that $z^2 = -1$ (the square of a real number is always positive).

Definition 38. Define an imaginary unit i such that

$$i^2 = -1.$$

More general, we define an imaginary number z to be of the form

$$z = iy, \ y \in RR$$

therefore

$$z^2 = (iy)^2 = i^2 y^2 = -y^2.$$

We may often use the notation $i = \sqrt{-1}$ to emphasize that i is not an index and is instead the imaginary unit, however, it is really best to avoid writing negative roots to avoid confusion in the computations...

Example 65. We have $(4i)^2 = 16i^2 = -16$.

Definition of i

- There is no real number z such that
$$z^2 = -1.$$

Define an imaginary unit i (denoted also j) such that
$$i^2 = -1 \ (that \ is) \ i = \sqrt{-1}.$$

Define an imaginary number z to be of the form
$$z = iy = y\sqrt{-1}$$
for y any real number.
- Then the imaginary number z is such that
$$z^2 = -y^2.$$

Computations with i

To avoid confusion, write i instead of a negative root!!

Powers of i: $\boxed{i^0 = 1, i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1.}$

Inverse of i: $i\,z = 1 \ thus \ z = \boxed{i^{-1} = -i.}$

Powers of z=iy, y real: $z^2 = i^2y^2 = -y^2, z^3 = -iy^3$

Example: $(4i)^2 = -16.$

Let us compute some powers of i:

$$i^0 = 1, \ i^1 = i, \ i^2 = -1, \ i^3 = i(i^2) = -i, \ i^4 = (i^2)^2 = (-1)^2 = 1.$$

To compute the inverse of i, we need to find an imaginary number z such that

$$iz = 1 \rightarrow z = i^{-1} = -i.$$

Correspondingly we get powers of imaginary numbers of the form $z = iy$, $y \in \mathbb{R}$:

$$z^0 = 1, \ z^1 = z, \ z^2 = (iy)^2 = -y^2, \ z^3 = iy(-y^2) = -iy^3, \ z^4 = (z^2)^2 = (-y^2)^2 = y^4.$$

Definition 39. We define a complex number z to be of the form

$$z = a + ib, \ a, b \in \mathbb{R}.$$

We call a, and write $\Re(z)$ the real part of z. We call b, and write $\Im(z)$ the imaginary part of z.

Example 66. Take $z = 3+5i$, then its real part is $\Re(z) = 3$ and its imaginary part is $\Im(z) = 5$.

Definition 40. We define the conjugate of a complex number $z = a + ib$, $a, b \in \mathbb{R}$, to be

$$\bar{z} = a - ib, \ a, b \in \mathbb{R}.$$

Note that for $z = a + ib$, $a, b \in \mathbb{R}$:

$$\bar{\bar{z}} = \overline{a - ib} = a + ib = z.$$

Also

$$z\bar{z} = (a + ib)(a - ib) = (a - ib)(a + ib) = a^2 + iab - iab - (i^2)b^2 = a^2 + b^2.$$

Example 67. Take $z = 3+5i$, then its conjugate is $\bar{z} = 3-5i$ and $z\bar{z} = 9+25$.

We discuss next how to visualize a complex number geometrically. To do so, we associate to a complex number $z = a + ib$ a pair comprising its real part and its imaginary part, namely (a, b). Now we see (a, b) in the two-dimensional real plane, where the real part corresponds to the x-axis, while the imaginary part corresponds to the y-axis.

Complex Numbers

A complex number z is of the form
$$z = a + bi, a, b \ real \ numbers.$$

We call a =Re(z) the real part of z, and b = Im(z) the imaginary part of z.

Example: 3+5i, Re(3+5i)=3, Im(3+5i)=5.

Conjugate

For z=a+ib a complex number, its conjugate $\bar{z}$ is
$$\bar{z} = a - ib.$$

- We have $\bar{\bar{z}}$=z.

- We have $z\bar{z} = \bar{z}z = a^2 + b^2$.

Example: $\overline{3 + 5i}$ =3-5i, $(\overline{3 + 5i})$(3+5i)=9+25.

We next define operations on complex numbers.

Addition. Take two complex numbers $a + ib$, $c + id$, their sum is given by

$$(a + ib) + (c + id) = (a + bc) + i(b + d).$$

Multiplication. Take two complex numbers $a + ib$, $c + id$, their product is given by

$$(a + ib)(c + id) = ac + aid + ibc - bd = (ac - bd) + i(ad + bc).$$

Division. Take two complex numbers $a + ib$, $c + id$, their ratio is given by

$$\frac{a + ib}{c + id}.$$

To be able to handle this case, the technique is to multiply both the numerator and denominator by the conjugate of the denominator, namely $c - id$:

$$\frac{a + ib}{c + id} = \frac{(a + ib)(c - id)}{(c + id)(c - id)} = \frac{(a + ib)(c - id)}{c^2 + d^2}.$$

Now we are in familiar territories since the denominator is a real number:

$$\frac{a + ib}{c + id} = \frac{(ac - iad + ibc + db)}{c^2 + d^2} = \frac{ac + db}{c^2 + d^2} + i\frac{bc - ad}{c^2 + d^2}.$$

We saw above that a complex number $z = a + ib$ can be represented as the point (a, b) in the 2-dimensional real plane. Recall that a point on a circle of radius one centered around the origin can be written as $(\cos\theta, \sin\theta)$, where θ is the angle from the x-axis counter clockwise. Now to be able to write similarly an arbitrary point in the 2-dimensional real plane, note that this point will be on a circle of radius r, where r is the length of the vector (a, b). Therefore, this point can be written as

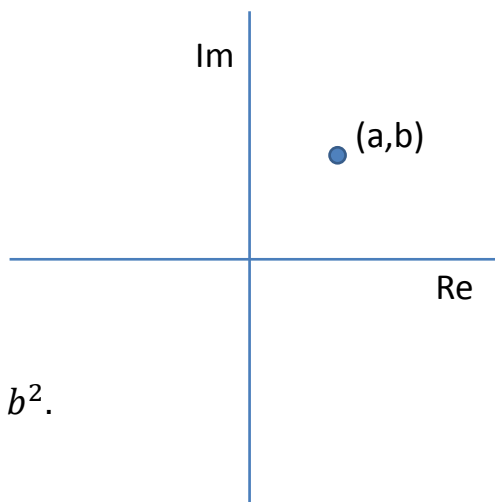$$(a, b) = (r\cos\theta, r\sin\theta),$$

which are called polar coordinates. Alternatively, we may write

$$z = a + ib = r\cos\theta + i\sin\theta.$$

We call r the modulus of z, and θ the argument of z.

Complex Plane

Geometrically:

a+ib, or (a,b)

Im

● (a,b)

Re

Also

$(\overline{a + ib})(a + ib) = a^2 + b^2.$

Complex Numbers Operations

Addition: $(a + ib) + (c + id) = (a + c) + i(b + d)$

Multiplication: $(a + ib)(c + id) = (ac - bd) + i(ad + bc)$

Division: $\dfrac{a+ib}{c+id} = \dfrac{(a+ib)(c-id)}{(c+id)\overline{(c+id)}}$

Once we have a complex number written in polar coordinates, we may use Euler formula to write the complex number in exponential form.

Euler formula says that

$$e^{i\theta} = \cos\theta + i\sin\theta$$

for θ any real number (in radians).

Proof. The proof that is provided now is the most classical one, and it relies on Taylor series for the different quantities involved:

$$
\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\
\sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \\
\cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots
\end{aligned}
$$

Recall that a Taylor series for a function $f(x)$ is

$$f(x) = f(a) + \frac{f'(a)}{1}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$
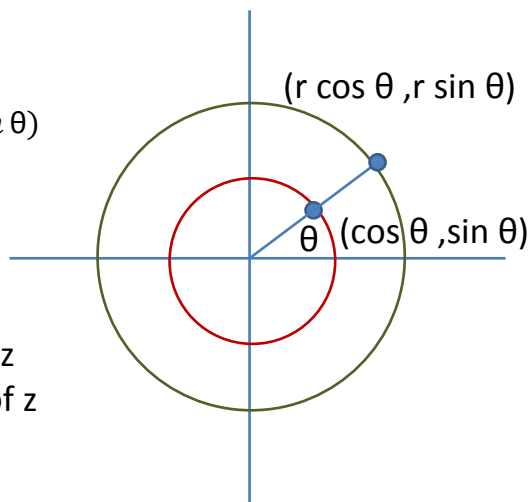
where a is taken to be zero in our case.

Thus

$$
\begin{aligned}
e^{ix} &= \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(ix)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(ix)^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} \frac{(i^2)^n}{(2n)!} x^{2n} + \sum_{n=0}^{\infty} \frac{i(i^2)^n}{(2n+1)!} x^{2n+1} \\
&= \cos x + i\sin x
\end{aligned}
$$

as needed. $\square$

Polar Coordinates

$$z = a + ib = r \left(\cos\theta + i\,\sin\theta\right)$$

(r cos θ ,r sin θ)

θ (cos θ ,sin θ)

r is called the modulus of z
θ is called the argument of z

Euler Formula

For θ any real number (in radians)
$$e^{i\theta} = \cos\theta + i\,\sin\theta.$$

(Recall that 2π radians =360 degrees)

Euler identity: $e^{i\pi} + 1 = 0.$

Leonhard Euler
(1707-1783)

Image from wikipedia

As a corollary, we get Euler identity:

$$e^{i\pi} + 1 = 0.$$

Indeed choose θ to be π (=180 degrees), then

$$e^{i\pi} = \cos\pi + i\sin\pi = -1,$$

since $\cos\pi = -1$ and $\sin\pi = 0$.

Thanks to Euler Formula, we can rewrite a complex number z as:

$$z = a + ib = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

Notice now that if we compute $z\bar{z}$, we get

$$z\bar{z} = (a + ib)(a - ib) = re^{i\theta}re^{-i\theta}$$

that is

$$z\bar{z} = a^2 + b^2 = r^2$$

therefore the modulus of z, denoted by $|z|$, satisfies

$$|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2} = r.$$

The argument (or phase) of z is θ, let us try to express it as a function of a, b. For that, remember that given a right triangle (drawn on the x-axis) with angle θ that goes from the x-axis counter clockwise, we have that $\tan\theta = \frac{b}{a}$ (the ratio of the opposite side and the adjacent one). Therefore $\theta = \tan^{-1}\frac{b}{a}$ where the range of tan^{-1} is $(-\pi, \pi]$, and one has to be careful that there are special cases depending on the sign of a, b:

$$\arg(z) = \begin{cases} \tan^{-1}\frac{b}{a} & a > 0 \\ \tan^{-1}\frac{b}{a} + \pi & a < 0, b \geq 0 \\ \tan^{-1}\frac{b}{a} - \pi & a < 0, b < 0 \\ \frac{\pi}{2} & a = 0, b > 0 \\ \frac{-\pi}{2} & a = 0, b < 0 \\ \text{indeterminate} & a = 0, b = 0. \end{cases}$$

Example 68. Take $z = 3 + 3\sqrt{3}i$, then its modulus $|z|$ is

$$|z| = \sqrt{3^2 + (3\sqrt{3})^2} = \sqrt{9 + 27} = 6.$$

Then for the phase

$$\arg z = \tan^{-1}\sqrt{3} = \frac{\pi}{3}.$$
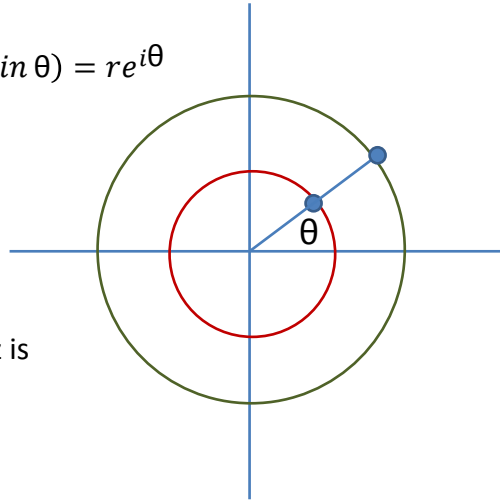
Converting among Forms

$$z = a + ib = r\left(\cos\theta + i\,\sin\theta\right) = re^{i\theta}$$

The modulus |z| of z is

$$|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2} = r.$$

The argument (or phase) of z is

$$\arg(z) = \theta = \tan^{-1}\frac{b}{a}.$$

Example

z=3+3√3 i

- $|z| = \sqrt{9 + 27} = 6.$

- $\arg(z) = \tan^{-1}\frac{3\sqrt{3}}{3} = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$

Thus the exponential form of z=3+3√3 i is $6e^{i\frac{\pi}{3}}$

while its polar form is: $6\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right).$

Thus in exponential form, we get:

$$z = 3 + 3\sqrt{3}i = 6e^{i\frac{\pi}{3}}$$

while the polar form is:

$$z = 3 + 3\sqrt{3}i = 6e^{i\frac{\pi}{3}} = 6\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

Definition 41. A root of unity is a complex number z such that $z^n = 1$.

Suppose we want to compute the 2nd roots of unity, we are looking for complex numbers z satisfying

$$z^2 = 1 \iff z^2 - 1 = 0 \iff (z-1)(z+1) = 0$$

and therefore there are two 2nd roots of unity: 1 and -1. Next we want to compute the 3rd roots of unity, we look at the equation

$$z^3 = 1 \iff z^3 - 1 = (z-1)(z^2 + z + 1) = 0.$$

Thus 1 is a root, and we need to identify the other two (it is a polynomial of degree 3, therefore we expect three roots). Consider $z = e^{2\pi i/3}$, then

$$z^3 = (e^{2\pi i/3})^3 = e^{2\pi i} = 1$$

and z is indeed a 3rd root of unity. So is $(e^{2\pi i/3})^2$ since

$$[(e^{2\pi i/3})^2]^3 = (e^{2\pi i})^2 = 1.$$

Note that apart 1, the other 2 roots are complex, which means that they do not exist in $\mathbb{R}$! (so one says that there are no roots in the case of real roots). In general, the n roots of $z^n = 1$ are

$$(e^{2\pi i/n})^k, \ k = 1, \dots, n.$$

This is because

$$[(e^{2\pi i/n})^k]^n = (e^{2\pi i})^k = 1$$

thus we have n distinct solutions, and so we got all of them!

We may ask the same question with another real number than 1, namely, what are the roots of $z^n = a$, for a a real number. The roots are

$$\sqrt[n]{a}(e^{2\pi i/n})^k, \ k = 1, \dots, n.$$

To check it, it is enough to compute the nth power, and see that we get indeed a:

$$(\sqrt[n]{a}(e^{2\pi i/n})^k)^n = a(e^{2\pi i})^k = a,$$

and we have found the n distinct roots.

Nth Roots of Unity

- What are the roots of $z^2 = 1$?

  1 and −1.

- What are the roots of $z^3 = 1$?

  $e^{\frac{2\pi i}{3}}, \left(e^{\frac{2\pi i}{3}}\right)^2, 1$

- What are the roots of $z^n = 1$?

  $\left(e^{\frac{2\pi i}{n}}\right)^k, k = 1, \ldots, n.$

Nth Roots

- What are the roots of $z^2 = 2$?

  $\sqrt{2}$ and $-\sqrt{2}$.

- What are the roots of $z^3 = 2$?

  $\sqrt[3]{2}e^{\frac{2\pi i}{3}}, \sqrt[3]{2}\left(e^{\frac{2\pi i}{3}}\right)^2, \sqrt[3]{2}$

- What are the roots of $z^n = a$, for a real?

  $\sqrt[n]{a}\left(e^{\frac{2\pi i}{n}}\right)^k, k = 1, \ldots, n.$

Exercises for Chapter 7

Exercise 54. Set $i = \sqrt{-1}$. Compute

$$i^5, \ \frac{1}{i^2}, \ \frac{1}{i^3}.$$

Exercise 55. Set $i = \sqrt{-1}$. Compute the real part and the imaginary part of

$$\frac{(1 + 2i) - (2 + i)}{(2 - i)(3 + i)}.$$

Exercise 56. Set $i = \sqrt{-1}$. Compute $d, e \in \mathbb{R}$ such that

$$4 - 6i + d = \frac{7}{i} + ei.$$

Exercise 57. For $z_1, z_2 \in \mathbb{C}$, prove that

- $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$.

- $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$.

Exercise 58. Consider the complex number z in polar form: $z = re^{i\theta}$. Express $re^{-i\theta}$ as a function of z.

Exercise 59. Prove that

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx,$$

for n an integer.

Exercise 60. Compute $|e^{i\theta}|$, $\theta \in \mathbb{R}$.

Exercise 61. Prove the so-called triangle inequality:

$$|a + b| \le |a| + |b|, \ a, b \in \mathbb{C}.$$

Exercise 62. Compute the two roots of $4i$, that is

$$\sqrt{4i}.$$