Chapter 9

Relations

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The topic of our next chapter is relations, it is about having 2 sets, and connecting related elements from one set to another.

Definition 53. Let A and B be two sets. A binary relation R from A to B is a subset of the cartesian product $A \times B$. Given $x, y \in A \times B$, we say that x is related to y by R, also written $(xRy) \leftrightarrow (x, y) \in R$.

Example 84. Suppose that you have two sets $A = \{1, 2\}$ and $B = \{1, 2, 3\}$, and the relation is given by $(x, y) \in R \leftrightarrow x - y$ is even. Since the relation is a subset of $A \times B$, we start by computing the cartesian product $A \times B$:

$$A \times B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}.$$

Then in this list of pairs, we select those which satisfy the relation R. For example, for $(1, 2)$, we have $x = 1$ and $y = 2$, we compute $x - y = 1 - 2 = -1$, which is odd, thus it does not belong to R. We try out similarly all the pairs in $A \times B$ to get

$$R = \{(1, 1), (1, 3), (2, 2)\}.$$

This may be visualized using a diagram: draw a circle to represent the set A, and this circle contains two points, one for 1 and one for 2. Similarly, draw a circle to represent B, and points of $1, 2, 3$. Then an arrow from A to B connects x in A with y in B if $x - y$ is even.

Binary Relations between Two Sets

Let A and B be sets. A binary relation R from A to B is a subset of $A \times B$. Given (x,y) in $A \times B$, x is related to y by R ($x \mathbf{R} y$) $\leftrightarrow (x,y) \in R$.
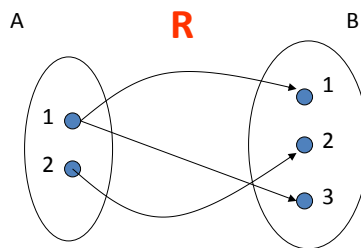
Example. $A=\{1,2\}$, $B=\{1,2,3\}$, $(x,y) \in R \leftrightarrow (x-y)$ is even.
- $A \times B = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)\}$
- $(1,1) \in R, (1,3) \in R, (2,2) \in R.$

Examples. x > y, x owes y, x divides y

Graphically

- **Example.** $A=\{1,2\}$, $B=\{1,2,3\}$, $(x,y) \in R \leftrightarrow (x-y)$ is even.
- $A \times B = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)\}$
- $(1,1) \in R, (1,3) \in R, (2,2) \in R.$

Definition 54. Let R be a relation from the set A to the set B. The inverse relation R^{-1} from B to A is defined as

$$R^{-1} = \{(y,x) \in B \times A, \ (x,y) \in R\}.$$

What it says is that for every pair (x,y) in R, you take it, flip the role of x and y to get (y,x), which then belongs to R^{-1}.

Example 85. Consider the sets $A = \{2,3,4\}$, $B = \{2,6,8\}$, with the relation $(x,y) \in R \leftrightarrow x$ divides y. Let us look at it step by step. First we compute the cartesian product $A \times B$:

$$A \times B = \{(2,2),(2,6),(2,8),(3,2),(3,6),(3,8),(4,2),(4,6),(4,8)\}.$$

Then we check for which pair (x,y) it is true that $x \mid y$. For example, if $(x,y) = (2,2)$, then $2 \mid 2$ and $(2,2) \in R$, but for $(x,y) = (3,2)$, 3 does not divide 2, and $(3,2)$ is not in R. Trying out all the pairs, we get

$$R = \{(2,2),(2,6),(2,8),(3,6),(4,8)\}.$$

Now for every pair $(x,y) \in R$, we flip the role of x and y to get

$$R^{-1} = \{(2,2),(6,2),(8,2),(6,3),(8,4)\}.$$

In this case, there is a nice interpretation of what R^{-1} means: $(x,y) \in R \leftrightarrow x|y$, but $x|y \iff y$ is a multiple of x and R^{-1} describes the relation $(y,x) \in R^{-1} \leftrightarrow y$ is a multiple of x. If one draws a diagram, then to go from R to R^{-1}, all is needed is to change the direction of the arrows!

Apart diagrams, another convenient way to represent a relation is to use a matrix representation. Take a binary relation R from the set $A = \{a_1, \ldots, a_m\}$ to the set $B = \{b_1, b_2, \ldots, b_n\}$. Create a matrix whose rows are indexed by the elements of A (thus m rows) and whose columns are indexed by the elements of B (thus n columns). Now the entry (i,j) of the matrix, corresponding to the ith row and jth column, contains $a_i R b_j$, that is, a truth value (True or False), depending on whether it is true or not that $a_i R b_j$ (that is, a_i is related to b_j).

Example 86. Take $A = \{2,3,4\}$, $B = \{2,6,8\}$ and the relation R defined by $(x,y) \in R \leftrightarrow x$ divides y. Then the rows of the matrix are indexed by $2,3,4$, and the columns by $2,6,8$. We thus get

$$\begin{pmatrix} 2R2 & 2R6 & 2R8 \\ 3R2 & 3R6 & 3R8 \\ 4R2 & 4R6 & 4R8 \end{pmatrix} = \begin{pmatrix} T & T & T \\ F & T & F \\ F & F & T \end{pmatrix}.$$

Inverse of a Binary Relation

Let R be a relation from A to B. The inverse relation R^{-1} from B to A is defined as: $R^{-1} = \{(y,x) \in B \times A \mid (x,y) \in R\}$.
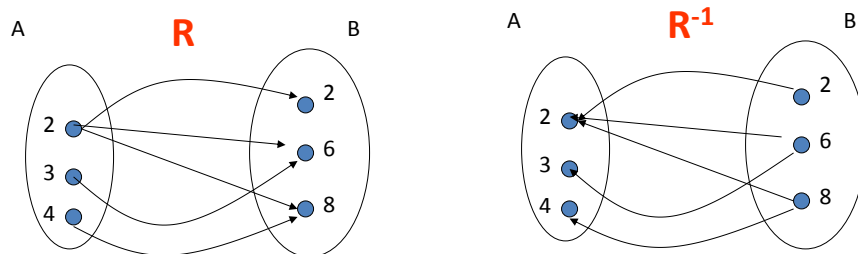
Example. $A=\{2,3,4\}$, $B=\{2,6,8\}$, $(x,y) \in R \leftrightarrow x$ divides y.
- $A \times B = \{(2,2),(2,6),(2,8),(3,2),(3,6),(3,8),(4,2),(4,6),(4,8)\}$
- $(2,2) \in R, (2,6) \in R, (2,8) \in R, (3,6) \in R, (4,8) \in R$

- $(2,2) \in R^{-1}, (6,2) \in R^{-1}, (8,2) \in R^{-1}, (6,3) \in R^{-1}, (8,4) \in R^{-1}$
- $(y,x) \in R^{-1} \leftrightarrow y$ is a multiple of x.

Graphically

Example. $A=\{2,3,4\}$, $B=\{2,6,8\}$, $(x,y) \in R \leftrightarrow x$ divides y.
- $(2,2) \in R, (2,6) \in R, (2,8) \in R, (3,6) \in R, (4,8) \in R$
- $(2,2) \in R^{-1}, (6,2) \in R^{-1}, (8,2) \in R^{-1}, (6,3) \in R^{-1}, (8,4) \in R^{-1}$

Matrix Representation (I)

$A = (a_1, a_2, a_3), B = (b_1, b_2, b_3, b_4,),$
$R = \{(a_1, b_2), (a_2, b_1), (a_3, b_1), (a_3, b_4)\}$

$a_i R b_j$ is represented by true, false else:
$$\begin{bmatrix} F & T & F & F \\ T & F & F & F \\ T & F & F & T \end{bmatrix}$$

Example. A={2,3,4}, B={2,6,8},
$(x, y) \in R \leftrightarrow x$ divides y.
$$\begin{bmatrix} T & T & T \\ F & T & F \\ F & F & T \end{bmatrix}$$

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Matrix Representation (II)

R relation from *A* to *B*: $R^{-1} = \{(y,x) \in B \times A \mid (x,y) \in R\}.$

$A = (a_1, a_2, a_3), B = (b_1, b_2, b_3, b_4,),$
$R = \{(a_1, b_2), (a_2, b_1), (a_3, b_1), (a_3, b_4)\}$
$R^{-1} = \{(b_2, a_1), (b_1, a_2), (b_1, a_3), (b_4, a_3)\}$

$a_i R b_j$ =true $\begin{bmatrix} F & T & F & F \\ T & F & F & F \\ T & F & F & T \end{bmatrix}$ $b_i R^{-1} a_j$=true $\begin{bmatrix} F & T & T \\ T & F & F \\ F & F & F \\ F & F & T \end{bmatrix}$

The matrix of R^{-1} is the transpose of the matrix of R.

Composition of Relations

Given R in $A \times B$, and S in $B \times C$, the composition of R and S is a relation on $A \times C$ defined by
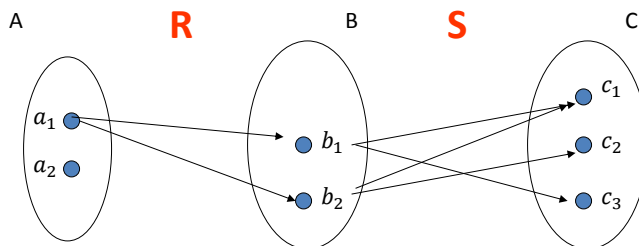$R \circ S = \{(a, c) \in A \times C \mid \exists\, b \in B, aRb \text{ and } bSc\}.$

Example. $A = \{a_1, a_2\}, B = \{b_1, b_2\}, C = \{c_1, c_2, c_3\}$
- $R = \{(a_1, b_1), (a_1, b_2)\}$
- $S = \{(b_1, c_1), (b_2, c_1), (b_1, c_3), (b_2, c_2)\}$
- What is $R \circ S$?

- $R \circ S = \{(a_1, c_1), (a_1, c_3), (a_1, c_2)\}$

Graphically

- **Example.** $A = \{a_1, a_2\}, B = \{b_1, b_2\}, C = \{c_1, c_2, c_3\}$
- $R = \{(a_1, b_1), (a_1, b_2)\}$
- $S = \{(b_1, c_1), (b_2, c_1), (b_1, c_3), (b_2, c_2)\}$
- $R \circ S = \{(a_1, c_1), (a_1, c_3), (a_1, c_2)\}$

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We may ask next how to interpret the inverse relation R^{-1} on its matrix. First of all, if R goes from $A = \{a_1, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$, then R^{-1} goes from B to A. This means that the rows of the matrix of R^{-1} will be indexed by the set $B = \{b_1, b_2, \ldots, b_n\}$, while its columns by the set $A = \{a_1, \ldots, a_m\}$. Then, by definition of R^{-1}, whenever there was a T (true) in row i and column j, this meant that $(a_i, b_j) \in R$, thus $(b_j, a_i) \in R^{-1}$, and this becomes a T (true) in row j and column i. If you take the first row of the matrix of R, whenever $(a_1, b_j) \in R$, for the column j, $(b_j, a_1) \in R^{-1}$, and a true in the first row of R becomes a true in the first column of R^{-1}, and the other entries which are false in the first row of R similarly become false in the first column of R^{-1}. This shows that the matrix of R^{-1} is the transpose of R! (recall that the transpose of a matrix is obtained by switching rows and columns).

Example 87. We continue the above example with $A = \{2, 3, 4\}$, $B = \{2, 6, 8\}$ and the relation R defined by $(x, y) \in R \leftrightarrow x$ divides y. We have that $R = \{(2, 2), (2, 6), (2, 8), (3, 6), (4, 8)\}$ thus $R^{-1} = \{(2, 2), (6, 2), (8, 2), (6, 3), (8, 4)\}$. Then the matrix of R and R^{-1} are respectively given by

$$\begin{pmatrix} T & T & T \\ F & T & F \\ F & F & T \end{pmatrix}, \quad \begin{pmatrix} T & F & F \\ T & T & F \\ T & F & T \end{pmatrix}.$$

We continue to explore properties of relations.

Definition 55. Given two relations $R \in A \times B$ and $S \in B \times C$, the composition of R and S is a relation on $A \times C$ defined by

$$R \circ S = \{(a, c) \in A \times C, \ \exists b \in B, \ aRb, bSc\}.$$

What it says is that for (a, c) to be part of your relation $R \circ S$, we need to find an element $b \in B$, with the property that a is in relation with b, and b is in relation with c. It is probably best visualize on a diagram: draw 3 circles for A, B, C, and arrows from A to B using the relation R, and arrows from B to C using the relation S. If you can find a path following those arrows from a to c, then (a, c) is in $R \circ S$.

Example 88. Consider the sets $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, $C = \{c_1, c_2, c_3\}$, with relations defined by

$$R = \{(a_1, b_1), (a_1, b_2)\}, \ S = \{(b_1, c_1), (b_2, c_1), (b_1, c_3), (b_2, c_2)\}.$$
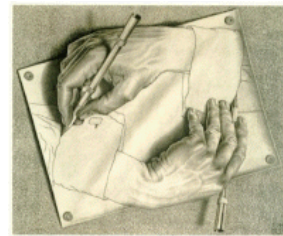
Reflexivity

A relation R on a set A is reflexive if every element of A is related to itself: $\forall x \in A, xRx$

Examples.
1. A=Z, xRy ⟷ x=y : reflexive

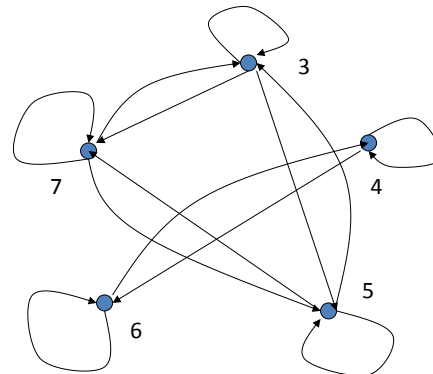2. A=Z, xRy ⟷ x>y : not reflexive

3. Reflexivity on the matrix
   representing R?

Escher, M C (1948): 'Drawing Hands'

Graphically

A= {3,4,5,6,7}, xRy ⟷ (x-y) is even

- *R reflexive*

To compute $R \circ S$, start with (a_1, b_1) and look for pairs starting with b_1 in S: (b_1, c_1) and (b_1, c_3). Therefore, (a_1, b_1) combined with (b_1, c_1) gives the pair (a_1, c_1) and the pair (a_1, b_1) combined with (b_1, c_3) gives (a_1, c_3). We do the same with (a_1, b_2) and pairs starting with b_2 in S to find (a_1, c_2), and

$$R \circ S = \{(a_1, c_1), (a_1, c_2), (a_1, c_3)\}.$$

So far, we were looking at binary relations from A to B. Next we focus on relations where $A = B$, that is we have relations from a set into itself.

Definition 56. A relation R on a set A is reflexive if every element of A is related to itself: $\forall x \in A, \ xRx$.

Example 89. If A is the set $\mathbb{Z}$ of integers, and the relation R is defined by $xRy \leftrightarrow x = y$, then this relation is reflexive, because it is true that x is always in relation with itself ($xRx \leftrightarrow x = x$ is always true).

But $xRy \leftrightarrow x > y$ is not reflexive, because it is never true that xRx (we never have $x > x$).

On the matrix representation of R, reflexivity is shown by having T (true) on the diagonal of the matrix. If one represents a relation on itself with a diagram, reflexivity will be seen by having arrows looping on every element of the diagram!

Definition 57. A relation R on a set A is symmetric if $(x, y) \in R$ implies $(y, x) \in R$: $\forall x \forall y \in A, \ xRy \rightarrow yRx$.

On a diagram, this is visualized with having a second arrow between 2 elements of A in the other direction whenever you have one arrow in one direction.

Example 90. If A is the set $\mathbb{Z}$ of integers, and the relation R is defined by $xRy \leftrightarrow x = y$, then this relation is symmetric, because it is true that if x is in relation with y then y is in relation with x ($xRy \leftrightarrow x = y$ implies $y = x \leftrightarrow yRx$).

But $xRy \leftrightarrow x > y$ is not symmetric, because it is never true that xRy implies yRx (we never have $x > y$ that implies $y > x$).

Definition 58. A relation R on a set A is transitive if $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$: $\forall x \forall y \forall z \in A, \ xRy \wedge yRz \rightarrow xRz$.
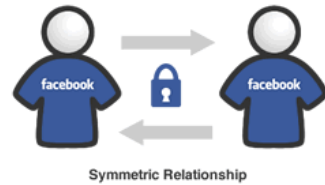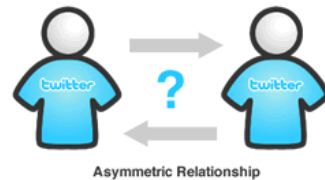
Symmetry

A relation R on a set A is symmetric if $(x,y) \in R$ implies $(y,x) \in R$: $\forall x \, \forall y \, xRy \rightarrow yRx$

Examples.

1. A=Z, xRy $\leftrightarrow$ x=y : symmetric

2. A=Z, xRy $\leftrightarrow$ x>y : not symmetric
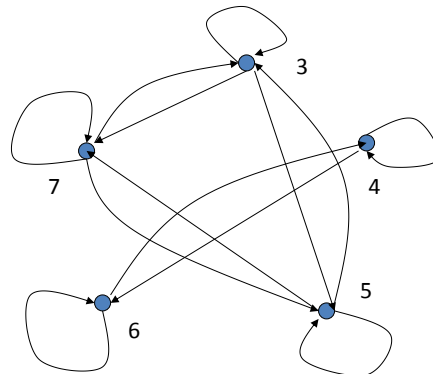

Asymmetric Relationship

Symmetric Relationship

http://bokardo.com/archives/relationship-symmetry-in-social-networks-why-facebook-will-go-fully-asymmetric/

Graphically

A= {3,4,5,6,7}, *xRy* $\leftrightarrow$ (*x-y*) is even

- *R reflexive*
- *R symmetric*

Example 91. If A is the set $\mathbb{Z}$ of integers, and the relation R is defined by $xRy \leftrightarrow x = y$, this relation is transitive, because it is true that if x is in relation with y and y is in relation with z then x is in relation with z ($xRy \leftrightarrow x = y$ and $y = z \leftrightarrow yRz$ implies that $x = y = z$ that is $x = z \leftrightarrow xRz$).

Also $xRy \leftrightarrow x > y$ is transitive, because if $xRy \leftrightarrow x > y$ and $yRz \leftrightarrow y > z$, then we have $x > y > z$ that is $x > z \leftrightarrow xRz$.

If a relation R on a set A turns out to satisfy the 3 properties we have just seen: reflexivity, symmetry, and transitivity, then this relation is special, and thus gets a special name:

Definition 59. A relation R on a set A is an equivalence relation if R is reflexive, symmetric and transitive. The equivalence class of a in A is

$$[a] = \{x \in A, \ aRx\}.$$

There is a reason for this name: an equivalence relation is so strong, it so strongly ties together elements that are in relation with each other, that instead of looking at elements one by one, we can just consider all those elements in relation with each other as one entity, called equivalence class.

Example 92. Consider the set $A = \{3, 4, 5, 6, 7\}$ with the relation $xRy \leftrightarrow (x - y)$ is even. Then R is reflexive: indeed, xRx is always true, since $(x - x) = 0$ which is even. Also R is symmetric: indeed, $xRy \leftrightarrow (x - y)$ is even implies that $-(x - y) = y - x$ is also even, and then $(y - x)$ is even $\leftrightarrow yRx$. Finally it is transitive: if $xRy \leftrightarrow (x - y)$ is even, and $yRz \leftrightarrow (y - z)$ is even, then $(x - z) = (x - y) + (y - z)$ which is even (sum of two even numbers is even), thus $(x - z)$ is even $\leftrightarrow xRz$. The equivalence class of [3] is the set of elements in relation with 3, that is $[3] = \{3, 5, 7\}$, similarly $[4] = \{4, 6\}$.

It turns out that equivalence classes partition A (for A a set with R a relation which is an equivalence relation). See Exercise 83.

The above example does form an equivalence relation, but it probably does not explain well the concept of equivalence relation, so let us try to get a better feeling using something that we already know (even though we do not know yet that these are equivalence classes!) namely, integers modulo n.

Transitivity

A relation R on a set A is transitive if $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$: $\forall x \; \forall y \; \forall z \; xRy \land yRz \rightarrow xRz$

Examples.

1. A=Z, xRy $\longleftrightarrow$ x=y : transitive

2. A=Z, xRy $\longleftrightarrow$ x>y : transitive

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http://www.apkdad.com/tag/atrium-for-facebook-apk/
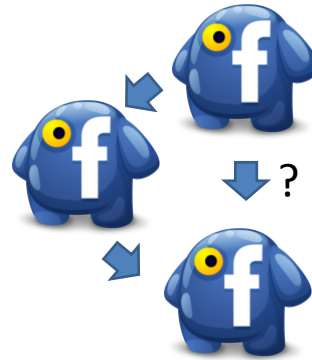
Equivalence Relation
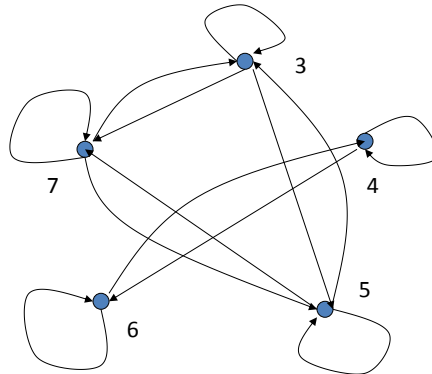
A relation R on a set A is an equivalence relation if

1. R is reflexive: $\forall x \; \in A, xRx$
2. R is symmetric: $\forall x \; \forall y \; xRy \; \rightarrow yRx$
3. R is transitive: $\forall x \; \forall y \; \forall z \; xRy \land yRz \rightarrow xRz$

Equivalence class of a in A: [a] ={x $\in A$ | aRx} for R an equivalence relation.

Example

A= {3,4,5,6,7}, *xRy* ⟷ (*x-y*) is even

- *R reflexive*

- *R symmetric*

- *R transitive*

- [3]={3,5,7},[4]={4,6}

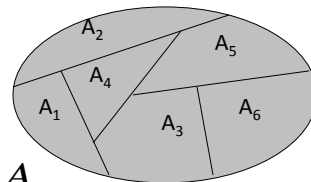# Equivalence Classes

Partition of a set A:

$$A_i \cap A_j = \phi \text{ whenever } i \neq j$$
$$A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6 = A$$

Equivalence classes of A form a partition of A.

Integers mod n (I)

$$a \equiv b \ (\text{mod } n) \longleftrightarrow a=qn+b$$

$\equiv$ (mod n) is an equivalence relation:

1. $\equiv$ (mod n) is reflexive: $\forall x \in A, x \equiv x \bmod n$
2. $\equiv$ (mod n) is symmetric: $\forall x \, \forall y \, x \equiv y \ (mod \ n) \rightarrow$ $y \equiv x \ (mod \ n)$
3. $\equiv$ (mod n) is transitive: $\forall x \, \forall y \, \forall z \, x \equiv$ y (mod n)$\wedge$ y$\equiv$z (mod n) $\rightarrow$ $x \equiv$z (mod n).

Integers mod n (II)

Equivalence class of [0]={0,n,2n,3n,…,-n,-2n,-3n…}
Equivalence class of [1]={1,n+1,2n+1,3n+1,…,-n+1,-2n+1…}

Example. Integers mod 4

- Integers mod n can be represented as elements between 0 and $n-1$: {0,1,2,…,n-1}

Example 93. The relation $\equiv$ (mod n) is an equivalence relation on $\mathbb{Z}$.

- It is reflexive: $x \equiv x$ (mod n) is always true.

- It is symmetric: $x \equiv y$ (mod n) means that $x = qn + y$ for some integer q, thus $y = -qn + x$ and $y \equiv x$ (mod n).

- It is transitive: if $x \equiv y$ (mod n) and $y \equiv z$ (mod n) then we have $x = qn + y$ and $y = rn + z$ thus $x = qn + y = qn + rn + z = n(q + r) + z$ and $x \equiv z$ (mod n).

Now what is the equivalence class of 0? it is formed by all multiples of n:

$$[0] = \{\ldots, -2n, -n, 0, n, 2n, \ldots\},$$

and similarly the equivalence class of 1 is all multiples of n, plus 1, and we see that there are exactly n equivalence classes, which partition $\mathbb{Z}$:

$$[0], [1], [2], \ldots, [n - 1].$$

This is why when we do operations modulo n, we are allowed to pick one element per equivalence class, namely $0, 1, \ldots, n - 1$ and work with them!!

We add one more property to those we know: reflexivity, symmetry, and transitivity.

Definition 60. A relation R on a set A is antisymmetric if $(x, y) \in R$ and $(y, x) \in R$ implies $x = y$: $\forall x \forall y, xRy \wedge yRx \rightarrow x = y$.

Note that symmetry and antisymmetric are not related, despite their name, see Exercise 80.

Example 94. If A is the set $\mathbb{Z}$ of integers, and the relation R is defined by $xRy \leftrightarrow x = y$, this relation is antisymmetric, because it is true that if x is in relation with y and y is in relation with x then $x = y$ ($xRy \leftrightarrow x = y$ and $y = x \leftrightarrow yRx$ implies that $x = y$).

Also $xRy \leftrightarrow x > y$ is antisymmetric, because we have a statement which is vacuously true!! if $xRy \leftrightarrow x > y$ and $yRx \leftrightarrow y > x$, well, this statement is always false...when we have a $p \rightarrow q$ where p is false then $p \rightarrow q$ is true (apply here with $p = "xRy \wedge yRx"$ and $q = "x = y"$).

Consider two sets B and C and the relation B is in relation with $C \leftarrow B \subseteq C$. Then $B \subseteq C$ and $C \subseteq B$ implies that $B = C$! this is what we used to show set equality (double inclusion), and this shows that this relation is antisymmetric!
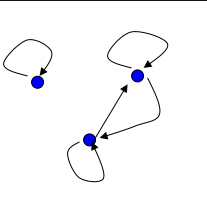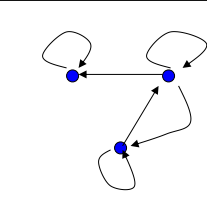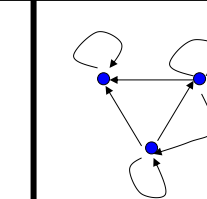
Antisymmetry

A relation R on a set A is antisymmetric if $(x, y) \in R$ and $(y, x) \in R$ implies x=y: $\forall x \, \forall y \; xRy \land yRx \; \rightarrow x = y$

Examples.
1. A=Z, xRy $\longleftrightarrow$ x=y : antisymmetric

2. A=Z, xRy $\longleftrightarrow$ x>y : vacuously true

3. BRC $\longleftrightarrow$ $B \subseteq C$: antisymmetric

Examples

Reflexive?	Y	Y	Y
Symmetric?	Y	N	N
Antisymmetric?	N	N	N
Transitive?	Y	N	Y

Definition 61. A relation R on a set A is a partial order if R is reflexive, antisymmetric and transitive.

The word partial order can be explained by the antisymmetry property. It is not possible to have "a loop" between two elements, namely a relation from one element to another, and back.

Example 95. If A is the set $\mathbb{Z}$ of integers, and the relation R is defined by $xRy \leftrightarrow x \leq y$, this relation is a partial order:

- It is reflexive: $x \leq x$ always.

- It is antisymmetric: $x \leq y$ and $y \leq x$ implies that $x = y$.

- It is transitive: if $x \leq y$ and $y \leq z$ then $x \leq y \leq z$ and thus $x \leq z$ as needed.

A set with a relation R may not satisfy the transitivity property, but then, one may wonder whether it is possible to "complete" the set with more elements to obtain the transitivity property. This gives rise to the notion of transitive closure:

Definition 62. Consider a relation R on a set A. The transitive closure of R is the binary relation R^t, that satisfies the properties:

- R^t is transitive,

- $R \subseteq R^t$,

- If S is any other transitive relation that contains R, then $R^t \subset S$.

The first property says the property of transitivity is satisfied, the second one that R is contained in R^t and the third one says R^t is minimal with this property!

Partial Order

A relation R on a set A is a partial order if R is reflexive, antisymmetric and transitive.

Example. A=Z, xRy $\longleftrightarrow$ x≤y

Notion of partial order useful for scheduling problems across possibly different domains.
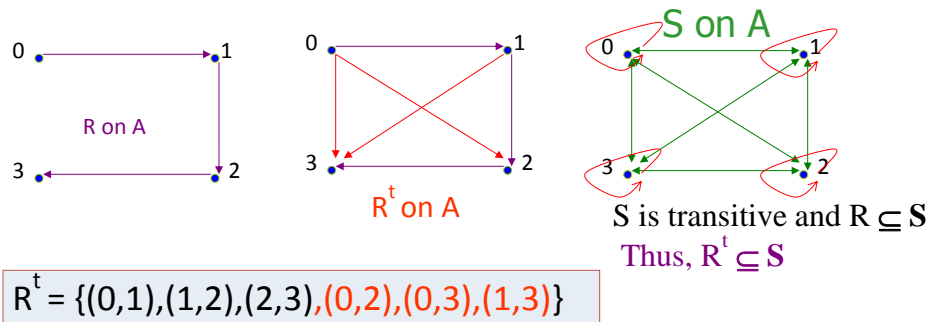
Transitive Closure

Let A be a set and R a binary relation on A.
The transitive closure of R is the binary relation R^t on A that satisfies the following three properties:

1. R^t is Transitive

2. $R \subseteq R^t$

3. If S is any other transitive relation that contains R, then $R^t \subseteq S$

Example

Let A = {0,1,2,3}
Consider a relation R = {(0,1),(1,2),(2,3)} on A.

R on A

R^t on A

S on A

S is transitive and R $\subseteq$ S
Thus, R^t $\subseteq$ S

R^t = {(0,1),(1,2),(2,3),(0,2),(0,3),(1,3)}

Non-binary Relations

Let $A_1, \ldots, A_n$ be sets. A n-ary relation R is a subset of $A_1 \times \cdots \times A_n$. $a_1, \ldots, a_n$ are related if $(a_1, \ldots, a_n) \in R$.

Exercises for Chapter 9

Exercise 72. Consider the sets $A = \{1, 2\}$, $B = \{1, 2, 3\}$ and the relation $(x, y) \in R \iff (x - y)$ is even. Compute the inverse relation R^{-1}. Compute its matrix representation.

Exercise 73. Consider the sets $A = \{2, 3, 4\}$, $B = \{2, 6, 8\}$ and the relation $(x, y) \in R \iff x \mid y$. Compute the matrix of the inverse relation R^{-1}.

Exercise 74. Let R be a relation from $\mathbb{Z}$ to $\mathbb{Z}$ defined by $xRy \leftrightarrow 2|(x - y)$. Show that if n is odd, then n is related to 1.

Exercise 75. This exercise is about composing relations.

1. Consider the sets $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, $C = \{c_1, c_2, c_3\}$ with the following relations R from A to B, and S from B to C:

$$R = \{(a_1, b_1), (a_1, b_2)\}, \quad S = \{(b_1, c_1), (b_2, c_1), (b_1, c_3), (b_2, c_2)\}.$$

   What is the matrix of $R \circ S$?

2. In general, what is the matrix of $R \circ S$?

Exercise 76. Consider the relation R on $\mathbb{Z}$, given by $aRb \iff a - b$ divisible by n. Is it symmetric?

Exercise 77. Consider a relation R on any set A. Show that R symmetric if and only if $R = R^{-1}$.

Exercise 78. Consider the set $A = \{a, b, c, d\}$ and the relation

$$R = \{(a, a), (a, b), (a, d), (b, a), (b, b), (c, c), (d, a), (d, d)\}.$$

Is this relation reflexive? symmetric? transitive?

Exercise 79. Consider the set $A = \{0, 1, 2\}$ and the relation $R = \{(0, 2), (1, 2), (2, 0)\}$. Is R antisymmetric?

Exercise 80. Are symmetry and antisymmetry mutually exclusive?

Exercise 81. Consider the relation R given by divisibility on positive integers, that is $xRy \leftrightarrow x|y$. Is this relation reflexive? symmetric? antisymmetric? transitive? What if the relation R is now defined over non-zero integers instead?

Exercise 82. Consider the set $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. Show that the relation $xRy \leftrightarrow 2|(x-y)$ is an equivalence relation.

Exercise 83. Show that given a set A and an equivalence relation R on A, then the equivalence classes of R partition A.

Exercise 84. Consider the set $A = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and the relation

$$xRy \leftrightarrow \exists c \in \mathbb{Z}, \ y = cx.$$

Is R an equivalence relation? is R a partial order?