Chapter 1

Isometries of the Plane

"For geometry, you know, is the gate of science, and the gate is so low and small that one can only enter it as a little child. (W. K. Clifford)

The focus of this first chapter is the 2-dimensional real plane $\mathbb{R}^2$, in which a point P can be described by its coordinates:

$$P \in \mathbb{R}^2,\ P = (x, y),\ x \in \mathbb{R},\ y \in \mathbb{R}.$$

Alternatively, we can describe P as a complex number by writing

$$P = (x, y) = x + iy \in \mathbb{C}.$$

The plane $\mathbb{R}^2$ comes with a usual distance. If $P_1 = (x_1, y_1),\ P_2 = (x_2, y_2) \in \mathbb{R}^2$ are two points in the plane, then

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Note that this is consistent with the complex notation. For $P = x + iy \in \mathbb{C}$, recall that $|P| = \sqrt{x^2 + y^2} = \sqrt{P\overline{P}}$, thus for two complex points $P_1 = x_1 + iy_1,\ P_2 = x_2 + iy_2 \in \mathbb{C}$, we have

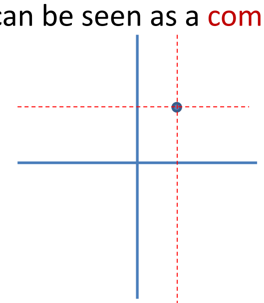
$$\begin{aligned} d(P_1, P_2) &= |P_2 - P_1| = \sqrt{(P_2 - P_1)\overline{(P_2 - P_1)}} \\ &= |(x_2 - x_1) + i(y_2 - y_1)| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}, \end{aligned}$$

where $\overline{(\ )}$ denotes the complex conjugation, i.e. $\overline{x + iy} = x - iy$.

We are now interested in planar transformations (that is, maps from $\mathbb{R}^2$ to $\mathbb{R}^2$) that preserve distances.

Points in the Plane

- A point P in the plane is a pair of real numbers P=(x,y).
  $d(0,P)^2 = x^2+y^2$.
- A point P=(x,y) in the plane can be seen as a complex number x+iy.
  $|x+iy|^2 = x^2+y^2$. $=d(0,P)^2$



Planar Isometries

An **isometry of the plane** is a transformation f of the plane that keeps distances unchanged, namely
$d(f(P_1),f(P_2)) = d(P_1,P_2)$
for any pair of points P_1,P_2.

- An isometry can be defined more generally than on a plane!

Definition 1. A map φ from $\mathbb{R}^2$ to $\mathbb{R}^2$ which preserves the distance between points is called a planar isometry. We write that

$$d(\varphi(P_1), \varphi(P_2)) = d(P_1, P_2)$$

for any two points P_1 and P_2 in $\mathbb{R}^2$.

What are examples of such planar isometries?

1. Of course, the most simple example is the identity map! Formally, we write
$$(x, y) \mapsto (x, y)$$
for every point $P = (x, y)$ in the plane.

2. We have the reflection with respect to the x-axis:
$$(x, y) \mapsto (-x, y).$$

3. Similarly, the reflection can be done with respect to the y-axis:
$$(x, y) \mapsto (x, -y).$$

4. Another example that easily comes to mind is a rotation.

Let us recall how a rotation is defined. A rotation counterclockwise through an angle θ about the origin $(0, 0) \in \mathbb{R}^2$ is given by

$$(x, y) \mapsto (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta).$$

This can be seen using complex numbers. We have that $|e^{i\theta}| = 1$, for $\theta \in \mathbb{R}$, thus

$$|(x + iy)e^{i\theta}| = |x + iy|$$

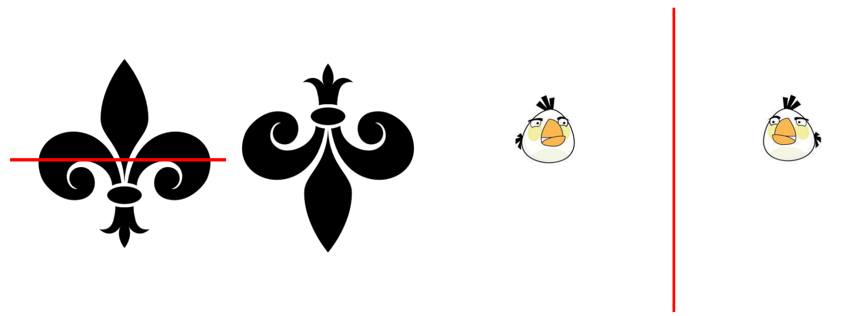
and multiplying by $e^{i\theta}$ does not change the length of (x, y). Now

$$\begin{aligned}(x + iy)e^{i\theta} &= (x + iy)(\cos\theta + i \sin\theta) \\ &= (x \cos\theta - y \sin\theta) + i(x \sin\theta + y \cos\theta)\end{aligned}$$

which is exactly the point $(x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$.

Examples of Isometries

- The identity map: (x,y) → (x,y)
- Mirror reflection w/r to the x-axis: (x,y) → (x,-y)
- Mirror reflection w/r to the y-axis : (x,y) → (-x,y)

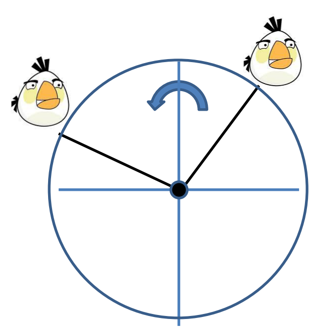


Angry Birds are owned by Rovio.

Rotation

- We also have a counterclockwise rotation of angle θ:

  (x,y) → (x cosθ-y sinθ, x sinθ+y cosθ)



In matrix notation, a rotation counterclockwise through an angle θ about the origin $(0,0) \in \mathbb{R}^2$ maps a point $P = (x, y)$ to $P' = (x', y')$, where $P' = (x', y')$ is given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{1.1}$$

We denote the rotation matrix by R_θ:

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Intuitively, we know that a rotation preserve distances. However, as a warm-up, let us prove that formally. We will give two proofs: one in the 2-dimensional real plane, and one using the complex plane.

First proof. Suppose we have two points $P_1 = (x_1, y_1),\ P_2 = (x_2, y_2) \in \mathbb{R}^2$. Let $d(P_1, P_2)$ be the distance from P_1 to P_2, so that the square distance $d(P_1, P_2)^2$ can be written as

$$\begin{aligned} d(P_1, P_2)^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ &= (x_2 - x_1, y_2 - y_1) \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} \\ &= \left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \right)^T \left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \right), \end{aligned}$$

where $()^T$ denotes the transpose of a matrix.

Now we map two points P_1, P_2 to P_1' and P_2' via (1.1), i.e.

$$\begin{bmatrix} x_i' \\ y_i' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} = R_\theta \begin{bmatrix} x_i \\ y_i \end{bmatrix}, \quad i = 1, 2.$$

Hence we have

$$\begin{bmatrix} x_2' \\ y_2' \end{bmatrix} - \begin{bmatrix} x_1' \\ y_1' \end{bmatrix} = R_\theta \left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \right),$$

and

$$\begin{aligned} d(P_1', P_2')^2 &= \left(\begin{bmatrix} x_2' \\ y_2' \end{bmatrix} - \begin{bmatrix} x_1' \\ y_1' \end{bmatrix} \right)^T \left(\begin{bmatrix} x_2' \\ y_2' \end{bmatrix} - \begin{bmatrix} x_1' \\ y_1' \end{bmatrix} \right) \\ &= \left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \right)^T R_\theta^T R_\theta \left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \right). \end{aligned}$$

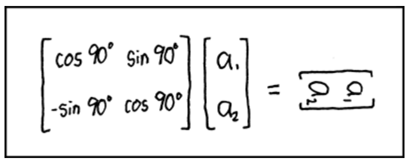
Rotations in Matrix Form

- If (x,y) is rotated counter-clockwise to get (x',y'), then

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Note: rotation around the origin!

where the rotation is written in matrix form.



Matrix transformation by xkcd

Rotations are Isometries : matrix proof

P_1 — P_2 rotate P_1' — P_2'

$P_1 = (x_1,y_1)$, $P_2 = (x_2,y_2)$, $P_1' = (x_1',y_1')$, $P_2' = (x_2',y_2')$

$$d(P_1,P_2)^2 \overset{?}{=} d(P_1',P_2')^2$$

$$d(P_1,P_2)^2 = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}^T \underbrace{R^T R}_{=\text{ identity matrix}} \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix} = \begin{bmatrix} x_2' - x_1' \\ y_2' - y_1' \end{bmatrix}^T \begin{bmatrix} x_2' - x_1' \\ y_2' - y_1' \end{bmatrix} = d(P_1',P_2')^2$$

R = rotation matrix we saw on the previous slide

But

$$R_\theta^T R_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which establishes that $d(P_1', P_2') = d(P_1, P_2)$.

Second proof. Let $P_1 = x_1 + iy_1,\ P_2 = x_2 + iy_2$ be two points in $\mathbb{C}$, with distance

$$d(P_1, P_2) = |P_2 - P_1| = \sqrt{(P_2 - P_1)\overline{(P_2 - P_1)}}.$$

Since a rotation of angle θ about the origin is represented by a multiplication by $e^{i\theta}$, we have

$$\begin{aligned} d(P_1', P_2') &= |P_2' - P_1'| = \left|e^{i\theta}P_2 - e^{i\theta}P_1\right| = \left|e^{i\theta}(P_2 - P_1)\right| \\ &= \left|e^{i\theta}\right| |P_2 - P_1| = |P_2 - P_1| = d(P_1, P_2). \end{aligned}$$

An arbitrary planar transformation maps $P = (x, y)$ to $P' = (\varphi(x, y), \psi(x, y))$, or in complex notation, $P = x + iy$ to $P' = \varphi(x, y) + i\psi(x, y) = H(P)$.

We are interested in special planar transformations, those which preserve distances, called isometries. We gave a few examples of planar isometries, we will next completely classify them.

To do so, we will work with the complex plane, and write an isometry as $H(z),\ z \in \mathbb{C}$, such that

$$|z_1 - z_2| = |H(z_1) - H(z_2)|\,.$$

We shall show that

Theorem 1. *If $|H(z_1) - H(z_2)| = |z_1 - z_2|$, for all $z_1, z_2 \in \mathbb{C}$, then $H(z) = \alpha z + \beta$ or $H(z) = \alpha\overline{z} + \beta$ with $|z| = 1$, i.e. $\alpha = e^{i\theta}$ for some θ.*

The theorem says that any function that preserves distances in $\mathbb{R}^2$ must be of the form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

or

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}.$$

Rotations are Isometries: complex proof

P_1 —— P_2 $\quad$ rotate $\quad$ P_1' —— P_2'

$$P_1 = X_1 + iy_1, \quad P_2 = x_2 + iy_2 \qquad P_1' = x_1' + iy_1', \quad P_2' = x_2' + iy_2'$$

$$d(P_1, P_2) \overset{?}{=} d(P_1', P_2')$$

$$d(P_1, P_2) = |P_2 - P_1| \qquad d(P_1', P_2') = |e^{\theta i}P_2 - e^{\theta i}P_1| = |e^{\theta i}|\,|P_2 - P_1|$$

$$|P_2 - P_1| = |e^{\theta i}P_2 - e^{\theta i}P_1|$$

Classification of Isometries of the plane

- Consider an arbitrary planar transformation map H, which maps a point P=x+iy to H(P).
- We are interested in classifying the maps H which are isometries, that is maps H satisfying $|H(z_1)-H(z_2)|=|z_1-z_2|$.

Notice what we recognize the reflections with respect to both the x- and y-axis, rotations around the origin, as well as translations.

In order to prove the theorem, we need the following cute lemma.

Lemma 1. *An isometry which maps* $(0,0)$ *to* $(0,0)$, $(1,0)$ *to* $(1,0)$, *and* $(0,1)$ *to* $(0,1)$, *i.e. (*0 *to* $0 \in \mathbb{C}$, 1 *to* $1 \in \mathbb{C}$, *and* i *to* $i \in \mathbb{C}$*) must be the identity map* $(x,y) \to (x,y)$.

Proof. The proof is done by identifying $\mathbb{R}^2$ with the complex plane. Let $h(z)$ be a planar isometry satisfying the assumptions of the lemma, in particular, $h(z)$ satisfies

$$|h(z_1) - h(z_2)| = |z_1 - z_2| \;\; \forall z_1, z_2 \in \mathbb{C}.$$

We then have

$$|h(z) - h(0)| = |z - 0|,$$

also

$$|h(z) - h(0)| = |h(z) - 0|$$

by assumption that $h(0) = 0$, thus

$$|h(z) - h(0)| = |h(z) - 0| = |z - 0|.$$

Using the fact that

$$h(1) = 1, \;\; h(i) = i,$$

we similarly get

$$\begin{aligned} |h(z) - 0| &= |h(z)| = |z - 0| = |h| \\ |h(z) - h(1)| &= |h(z) - 1| = |z - 1| \\ |h(z) - h(i)| &= |h(z) - i| = |z - i|\,. \end{aligned}$$

This shows that

$$\begin{aligned} &h(z)\overline{h(z)} = z\overline{z} \\ &(h(z) - 1)\overline{(h(z) - 1)} = (z - 1)\overline{(z - 1)} \\ &(h(z) - i)\overline{(h(z) - i)} = (z - i)\overline{(z - i)}. \end{aligned}$$

We now multiply out

$$(h(z)-1)(\overline{h(z) - 1}) = h(z)\overline{h(z)} - h(z) - \overline{h(z)} + 1 = (z-1)\overline{(z - 1)} = z\bar{z} - z - \bar{z} + 1,$$

A Lemma (I)

<u>Lemma</u> An isometry which maps 0 to 0, 1 to 1 and i to i must be the identity map.

<u>Proof</u>

Let H be an isometry: $|H(z_1)-H(z_2)|^2=|z_1-z_2|^2$ for every z_1, z_2.

By assumption H(0)=0, H(1)=1, H(i)=i.

1) $z\bar{z} = |z|^2 \overset{\text{H isometry}}{=} |H(z)-H(0)|^2 \overset{\text{H(0)=0}}{=} |H(z)|^2 = H(z)\overline{H(z)}$

2) $(z-1)(\overline{z-1}) = |z-1|^2 \overset{\text{H isometry}}{=} |H(z)-H(1)|^2 \overset{\text{H(1)=1}}{=} |H(z)-1|^2 = (H(z)-1)(\overline{H(z)-1})$

3) $(z-i)(\overline{z-i}) = |z-i|^2 \overset{\text{H isometry}}{=} |H(z)-H(i)|^2 \overset{\text{H(i)=i}}{=} |H(z)-i|^2 = (H(z)-i)(\overline{H(z)-i})$

A Lemma (II)

<u>Proof (next)</u>

From 2) : $H(z)\overline{H(z)} - H(z) - \overline{H(z)} + 1 = z\bar{z} - z - \bar{z} + 1 \rightarrow H(z)+\overline{H(z)} = z+\bar{z}$

From 3) : $H(z)\overline{H(z)} + iH(z) - i\overline{H(z)} + 1 = z\bar{z} + zi - i\bar{z} + 1 \rightarrow H(z) - \overline{H(z)} = z - \bar{z}$

We sum the last two equations to get H(z)=z.

QED

A point P which is fixed by a transformation f of the plane , that is a point such that f(P)=P is called a **fixed point**.

which can be simplified using that $h(z)\overline{h(z)} = z\overline{z}$, and similarly multiplying out $(h(z)-i)\overline{(h(z)-i)} = (z-i)\overline{(z-i)}$, we obtain

$$\begin{aligned} h(z)+\overline{h(z)} &= z+\overline{z} \\ h(z)-\overline{h(z)} &= z-\overline{z}. \end{aligned}$$

By summing both equations, we conclude that $h(z) = z$. □

In words, we have shown that if $h(z)$ has the same distances to $0, 1, i$ as z then $h(z)$ and z must be the same. This technique of looking at points which are fixed by a given planar transformation is useful and we will see it again later. It is thus worth giving a name to these special fixed points.

Definition 2. Let φ be a planar transformation. Then a point P in the plane such that $\varphi(P) = P$ is called a fixed point of φ.

We are now ready to classify planar isometries, that is to prove Theorem 1.

Proof. Given $H(z)$, an isometry $H : \mathbb{C} \to \mathbb{C}$, define

$$\begin{aligned} \beta &= H(0), \\ \alpha &= H(1)-H(0) \\ (|\alpha| &= |H(1)-H(0)| = |1-0| = 1). \end{aligned}$$

Now consider a new function

$$K(z) = \frac{H(z)-H(0)}{H(1)-H(0)} = \alpha^{-1}(H(z)-\beta).$$

Note the denominator is non-zero! Claim: $K(z)$ is also an isometry. Indeed, for every $z, w \in \mathbb{C}$, we have

$$\begin{aligned} |K(z)-K(w)| &= \left|\frac{H(z)-\beta}{\alpha} - \frac{H(w)-\beta}{\alpha}\right| \\ &= \left|\frac{H(z)-H(w)}{\alpha}\right| = \frac{|H(z)-H(w)|}{|\alpha|} \\ &= |H(z)-H(w)| = |z-w|. \end{aligned}$$

Now

$$\begin{aligned} K(0) &= \frac{H(0)-H(0)}{H(1)-H(0)} = 0 \\ K(1) &= \frac{H(1)-H(0)}{H(1)-H(0)} = 1. \end{aligned}$$

Main Result (I)

Theorem An isometry H of the complex plane is necessarily of the form

- $H(z)=\alpha z+\beta$, or
- $H(z)=\alpha\bar{z}+\beta$

with $|\alpha|=1$ and some complex number β.

Proof Given H an isometry, define

- $\beta=H(0)$
- $\alpha=H(1)-H(0)$

Theorem statement claims $|\alpha|=1$, needs a check!

Note that $|\alpha|=|H(1)-H(0)|=|1-0|=1$ as stated.

H isometry

Main Result (II)

- Consider a new function $K(z)=(H(z)-H(0))/(H(1)-H(0))$

$\beta=H(0)$, $\alpha=H(1)-H(0)$

- We have $K(z)=\alpha^{-1}(H(z)-\beta)$

$|\alpha|=1$ H isometry

- K(z) is an isometry:
  $|K(z)-K(w)| = |\alpha^{-1}||(H(z)-\beta)-(H(w)-\beta)| = |H(z)-H(w)|=|z-w|$.

Then

$$\begin{aligned} |K(i)| &= |i| = 1 \\ |K(i) - 1| &= |i - 1| = \sqrt{2}. \end{aligned}$$

These two equations tell us that $K(i)$ is either i or $-i$. This can be seen from a geometric point of view, by noticing that $K(i)$ is both on the unit circle around the origin 0 and on a circle of radius $\sqrt{2}$ around 1. Alternatively, multiplying out $(K(i) - 1)\overline{(K(i) - 1)} = 2$ and simplifying the expression obtained with $K(i)\overline{K(i)} = 1$ leads to the same conclusion.

If $K(i) = i$, then by Lemma 1, we have that

$$K(z) = z \Rightarrow H(z) = \alpha z + \beta.$$

If instead $K(i) = -i$, then $\overline{K(z)}$ is an isometry that fixes $0, 1, i$ hence

$$\overline{K(z)} = z \Rightarrow K(z) = \overline{z},\ \forall z \in \mathbb{C},$$

and in this case

$$H(z) = \alpha \overline{z} + \beta.$$

□

Let us stare at the statement of the theorem we just proved for a little bit. It says that every planar isometry has a particular form, and we can recognize some of the planar isometries that come to our mind (rotations around the origin, reflections around either the x- and y-axis, translations,...). But then, since we cannot think of other transformations, does it mean that no other exists? One can in fact prove the following:

Theorem 2. *Any planar isometry is either*

1. *a pure translation,*
2. *a pure rotation about some center z_0,*
3. *a reflection about a general line,*
4. *a glide reflection (that is, a reflection followed by a translation).*

We will come back to this theorem later!

Main Result (III)

- $K(0)=\alpha^{-1}(H(0)-\beta)=0$
- $K(1)=\alpha^{-1}(H(1)-\beta)=1$
- $K(i)=?$
- We know: $|K(i)|=|i|=1$
- We also know $|K(i)-1|=|i-1|=\sqrt{2}$

$\Rightarrow$ $K(i)=i$ or $-i$.

Main Result (IV)

- If $K(i)=i$, then by the previous lemma, we know that $K(z)=z$.
- $K(z)=\alpha^{-1}(H(z)-\beta)=z \longrightarrow H(z)=\alpha z+\beta$
- If $K(i)=-i$, then $\overline{K(i)}=i,\quad \overline{K(1)}=1, \overline{K(0)}=0$
- Also $|\overline{K(z)}-\overline{K(w)}|=|K(z)-K(w)|=|z-w|$
- Again by the previous lemma, we know that $\overline{K(z)}=z$
- Equivalently : $K(z)=\bar{z}$
- $K(z)=\alpha^{-1}(H(z)-\beta)=\bar{z}$

$\longrightarrow$ $H(z)=\alpha\bar{z}+\beta$.

QED

Next we shall show an easy consequence.

Theorem 3. *Any planar isometry is invertible.*

Proof. We check by direct computation that both possible formulas for isometries, namely

$$H(z) = \alpha z + \beta \text{ and } H(z) = \alpha \overline{z} + \beta,\ \alpha = e^{i\theta}, \beta \in \mathbb{C}$$

are invertible. If $z' = H(z) = \alpha z + \beta$, then

$$z = H^{-1}(z') = \frac{z' - \beta}{\alpha} = e^{-i\theta}(z' - \beta).$$

If instead $z' = H(z) = \alpha \overline{z} + \beta$, then

$$\overline{z} = \frac{z' - \beta}{\alpha} = e^{-i\theta}(z' - \beta)$$

and

$$z = H^{-1}(z') = \overline{e^{-i\theta}}(\overline{z'} - \overline{\beta}).$$

□

Remark. It is important to note that we have shown that a planar isometry is a bijective map. In general, one can define an isometry, but if it is not planar (that is, not from $\mathbb{R}^2$ to $\mathbb{R}^2$), then the definition of isometry usually includes the requirement that the map is bijective by definition. Namely a general isometry is a bijective map which preserves distances.

We now show that we can compose isometries, i.e. apply them one after the other, and that the result of this combination will yield another isometry, i.e., if H_1 and H_2 are two isometries then so is $H_2 H_1$.

Here are two ways of doing so.

First proof. We can use the definition of planar isometry. We want show that $H_2 H_1$ is an isometry. We know that

$$|H_2(H_1(z)) - H_2(H_1(w))| = |H_1(z) - H_1(w)|,$$

because H_2 is an isometry, and furthermore

$$|H_1(z) - H_1(w)| = |z - w|,$$

this time because H_1 is an isometry. Thus

$$|H_2(H_1(z)) - H_2(H_1(w))| = |z - w|,$$

for any $z, w \in \mathbb{C}$ which completes the proof.

Corollary

Corollary Any planar isometry is invertible.

Proof We know by the theorem: every isometry H is of the form

- $H(z)= \alpha z + \beta$, or
- $H(z)=\alpha \bar{z} + \beta$.

Let us compute H^{-1} in the first case.

- Define $H^{-1}(y)=(y-\beta)\alpha^{-1}$
- Check! $H(H^{-1}(y))=H((y-\beta)\alpha^{-1})=y$.
- Other case is done similarly!

QED

Combining Isometries

- The composition of two isometries is again an isometry!
- Let H and F be two isometries, then F(H(z)) is the composition of F and H.
- We have $|F(H(z))-F(H(w))|=|H(z)-H(w)|=|z-w|$.

F isometry H isometry

Second proof. Alternatively, since H_1, H_2 both have two types (we know that thanks to Theorem 1), there are 4 cases to be verified.

1. $H_2(H_1(z)) = \alpha_2(\alpha_1 z + \beta_1) + \beta_2 = (\alpha_2\alpha_1)z + (\alpha_2\beta_1 + \beta_2)$,
2. $H_2(H_1(z)) = \alpha_2(\alpha_1 \overline{z} + \beta_1) + \beta_2 = (\alpha_2\alpha_1)\overline{z} + (\alpha_2\beta_1 + \beta_2)$,
3. $H_2(H_1(z)) = \alpha_2(\overline{\alpha_1 z} + \overline{\beta_1}) + \beta_2 = (\alpha_2\overline{\alpha_1})\overline{z} + (\alpha_2\overline{\beta_1} + \beta_2)$,
4. $H_2(H_1(z)) = \alpha_2(\overline{\alpha_1} z + \overline{\beta_1}) + \beta_2 = (\alpha_2\overline{\alpha_1})z + (\alpha_2\overline{\beta_1} + \beta_2)$.

Note that isometries do not commute in general, that is

$$H_2(H_1(z)) \neq H_1(H_2(z))$$

since for example $\alpha_2\beta_1 + \beta_2 \neq \alpha_1\beta_2 + \beta_1$.

But we do have associativity, i.e.

$$H_3(H_2(H_1(z))) = (H_3H_2)(H_1(z)) = H_3(H_2H_1(z)).$$

We also see that the identity map $1 : z \mapsto 1(z) = z$ is an isometry, and when any planar isometry H is composed with its inverse, we obtain as a result the identity map 1:

$$\begin{aligned} H(H^{-1}(z)) &= 1(z) \\ H^{-1}(H(z)) &= 1(z). \end{aligned}$$

What we have proved in fact is that planar isometries form a set of maps which, together with the natural composition of maps, have the following properties:

1. associativity,
2. existence of an identity map (that is a map 1 such that when combined with any other planar isometry H does not change H: $H(1(z)) = 1(H(z)) = z$),
3. inverse for each map.

As we shall see later, this proves that the set of isometries together with the associative binary operation of composition of isometries is a **group**.

Exercises for Chapter 1

Exercise 1. Let X be a metric space equipped with a distance d. Show that an isometry of X (with respect to the distance d) is always an injective map.

Exercise 2. Show that an isometry of the complex plane that fixes three non-colinear points must be the identity map.

Exercise 3. In this exercise, we study the fixed points of planar isometries.

1. Recall the general formula that decribes isometries H of the complex plane.

2. Determine the fixed points of these transformations. Discuss the various cases that arise.