Chapter 2

Symmetries of Shapes

"Symmetries delight, please and tease !" (A.M. Bruckstein)

In the previous chapter, we studied planar isometries, that is maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ that are preserving distances. In this chapter, we will focus on different sets of points in the real plane, and see which planar isometries are preserving them.

We are motivated by trying to get a *mathematical formulation* of what is a "nice" regular geometric structure. Intuitively we know of course! We will see throughout this lecture that *symmetries* explain mathematically the geometric properties of figures that we like.

Definition 3. A symmetry of a set of points S in the plane is a planar isometry that preserves S (that is, that maps S to itself).

Note that "symmetries" also appear with letters and numbers! For example, the phrase

$$NEVER\ ODD\ OR\ EVEN$$

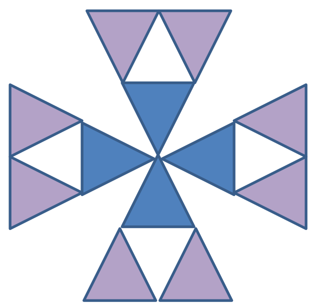
reads the same backwards! It is called a **palindrome**.

The same holds for the number 11311 which happens to be a prime number, called a **palindromic prime**.

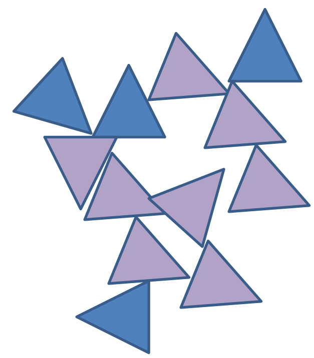
Palindromes can be seen as a conceptual mirror reflection with respect to the vertical axis, which sends a word to itself.

What is structure?

One intuitively knows …
that this is structured…



and this is random.



Symmetry

A **symmetry of a set of points S** is a planar isometry that preserves the set S (that is, that maps S to itself).

Among planar isometries, which can be symmetries of **finite** sets?

- ~~Translations~~
- Rotations
- Reflections
- The identity map!
- **Combinations of the above**

Recall from Theorem 2 that we know all the possible planar isometries, and we know the composition of planar isometries is another planar isometry! All the sets of points that we will consider are finite sets of points centered around the origin, thus we obtain the following list of possible symmetries:

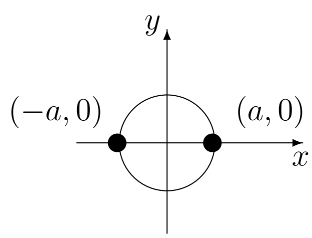
- the trivial identity map $1 : (x, y) \mapsto (x, y)$,
- the mirror reflections $m_v : (x, y) \mapsto (-x, y)$, $m_h : (x, y) \mapsto (-x, y)$ with respect to the y-axis, respectively x-axis, and in fact any reflection around a line passing through the origin,
- the rotation r_ω about 0 counterclockwise by an angle ω

$$\begin{aligned} r_\omega : (x, y) &\mapsto \begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= (x\cos\omega - y\sin\omega, x\sin\omega + y\cos\omega)\,. \end{aligned}$$

Translations are never possible! Consider first the set of points

$$S = \{(a, 0), (-a, 0)\}$$

(shown below) and let us ask what are the symmetries of S.



Clearly the **identity map** is one, it is a planar isometry and $1S = S$. The **mirror reflection** m_v with respect to the y-axis is one as well, since m_v is a planar isometry, and

$$m_v(a, 0) = (-a, 0),\ m_v(-a, 0) = (a, 0) \Rightarrow m_v(S) = S,$$

that is S, is **invariant** under m. Now choosing $\omega = \pi$, we have

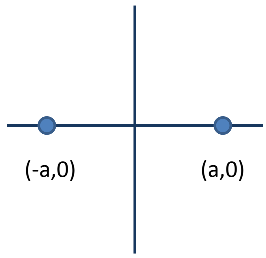
$$r_\pi(x, y) = (x\cos\pi - y\sin\pi, x\sin\pi + y\cos\pi) = (-x, -y)\,,$$

and

$$r_\pi(a, 0) = (-a, 0),\ r_\pi(-a, 0) = (a, 0) \Rightarrow r_\pi(P) = m_v(P)$$

for both points $P \in S$, which shows formally that rotating counterclockwise these two points by π about 0 is the same thing as flipping them around the y-axis.

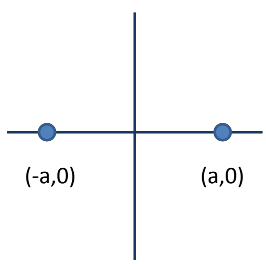
Symmetries of Two Aligned Points (I)



Consider the set of points
S={(a,0),(-a,0)}.
What are its symmetries?

1. The identity map 1 is a trivial symmetry of S!
2. Reflection m_v with respect to the y-axis
(a,0)→(-a,0),(-a,0)→(a,0)

Symmetries of Two Aligned Points (II)



Have we found all its symmetries?

YES!

Combining these symmetries **does not** give a **new** symmetry! We summarize these symmetries using a **multiplication table**.

	1	m
1	1	m
m	m	$1=m^2$

We have identified that the set $S = \{(a, 0), (-a, 0)\}$ has 2 symmetries. These are 1 and m_v, or 1 and r_π. We know that planar isometries can be composed, which yields another planar isometry. Then symmetries of S can be composed as well, and here we might wonder what happens if we were to compose m_v with itself:

$$m_v(m_v(x, y)) = m_v(-x, y) = (x, y)$$

which shows that $m_v(m_v(x, y)) = 1(x, y)$. We summarize the symmetries of $S = \{(a, 0), (-a, 0)\}$ using a **multiplication table**:

	1	m_v
1	1	m_v
m_v	m_v	$1 = m_v^2$

The multiplication table is read from left (elements in the column) to right (elements in the row) using as operation the composition of maps.

Let us collect what we have done so far. We defined a set of points $S = \{(a, 0), (-a, 0)\}$ and we looked at three transformations $1, m_v$ and r_π which leave the set of points of $S \in \mathbb{R}^2$ invariant:

$$\begin{cases} 1S = S \\ m_v S = S \\ r_\pi S = S \end{cases} \tag{2.1}$$

We saw that for this particular choice of S, we have that $r_\pi(P) = m_v(P)$ for both points $P \in S$.

The transformations are however different if we look at a "test point" $(x_0, y_0) \notin S$

$$\begin{cases} 1(x_0, y_0) \to (x_0, y_0) \\ m(x_0, y_0) \to (-x_0, y_0) \\ r_\pi(x_0, y_0) \to (-x_0, -y_0) \end{cases}$$

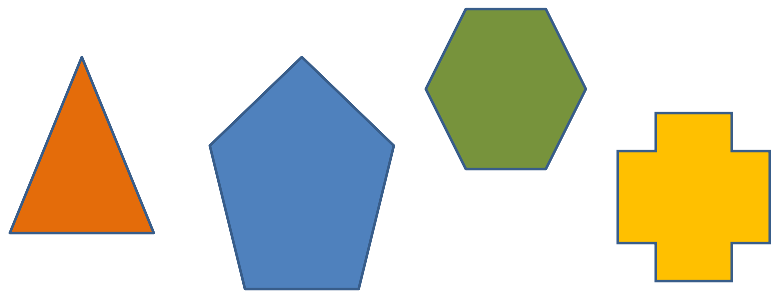
In fact, one may wonder what happens if we choose for S other sets of points, for example, different polygons. As our next example, we will look at a rectangle S. We write the rectangle S as

$$S = \{(a, b), (-a, b), (-a, -b), (a, -b)\}, \ a \neq b, \ a, b \neq 0. \tag{2.2}$$

(It is important that $a \neq b$! see (2.3 if $a = b$).)

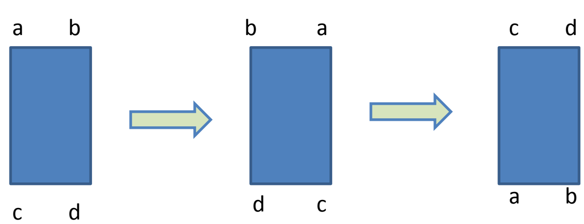
Symmetries of different shapes…

- Let us start with geometric objects:



Symmetries of the Rectangle (I)

- Let m be the vertical mirror reflection.
- Let r be a rotation of 180 degrees.
- Let 1 be the do-nothing symmetry.
- What is rm?



This is the horizontal mirror reflection!

Let us apply m_v on S:

$$m_v(a,b) = (-a,b),\ m_v(-a,b) = (a,b),$$
$$m_v(-a,-b) = (a,-b),\ m_v(a,-b) = (-a,-b)$$

as well as r_π:

$$r_\pi(a,b) = (-a,-b),\ r_\pi(-a,b) = (a,-b),$$
$$r_\pi(-a,-b) = (a,b),\ r_\pi(a,-b) = (-a,b).$$

These two maps are different and have different effects on S since $r_\pi(a,b) = (-a,-b) \neq (-a,b) = m_v(a,b)$. We now try to compose them. We already have $m_v(m_v(x,y)) = 1(x,y)$, and

$$r_\pi\left(r_\pi(x,y)\right) = r_\pi(-x,-y) = (x,y) = 1(x,y).$$

We continue with

$$r_\pi\left(m_v(x,y)\right) = r_\pi(-x,y) = (x,-y),\ m_v\left(r_\pi(x,y)\right) = m_v(-x,-y) = (x,-y)$$

which both give **a horizontal mirror reflection** m_h, also showing that

$$r_\pi m_v = m_v r_\pi = m_h,$$

i.e., the transformations r_π and m_v commute. In turn, we immediately have

$$(r_\pi m_v)^2 = r_\pi m_v r_\pi m_v = r_\pi m_v m_v r_\pi = r_\pi 1 r_\pi = r_\pi r_\pi = 1.$$

The rules for combining elements from $\{1, m_v, r_\pi, m_v r_\pi\}$

$$\begin{cases} m_v 1 = m_v = 1 m_v \\ r_\pi 1 = r_\pi = 1 r_\pi \\ m_v^2 = 1 \\ r_\pi^2 = 1 \\ m_v r_\pi = r_\pi m_v \end{cases}$$

show that no new transformations will ever be obtained since we have
$r_\pi^{(\alpha_i)} = r_\pi^{\alpha_i \mod 2}$, $m_v^{(\beta_i)} = m_v^{\beta_i \mod 2}$, $\pi^{\alpha_1} m_v^{\beta_1} r_\pi^{\alpha_2} m_v^{\beta_2} \cdots = r_\pi^{(\sum \alpha_i) \mod 2} m_v^{(\sum \beta_i) \mod 2}$.

Hence we have obtained a complete set of transformations for the shape S summarized in its multiplication table (we write $m = m_v$ for short):

	1	m	r_π	mr_π
1	1	m	r_π	mr_π
m	m	1	mr_π	r_π
r_π	r_π	mr_π	1	m
mr_π	mr_π	r_π	m	1

Symmetries of the Rectangle (II)

We thus have identified 4 symmetries:

- 1=the identity map
- m=vertical mirror reflection
- r=rotation of 180 degrees
- rm=horizontal mirror reflection

Note that

- $m^2=1$
- $r^2=1$
- $(rm)^2=1$
- rm=mr

Symmetries of the Rectangle (III)

	1	r	m	rm
1	1	r	m	rm
r	r	1	rm	m
m	m	rm	1	r
rm	rm	m	r	1

We next study the symmetries of a square, that is we consider the set

$$S_4 : \{(a,a), (-a,a), (a,-a)(-a,-a)\} \tag{2.3}$$

(this is the case where $a = b$ in (2.2)).

As for the two previous examples, we first need to see what are all the planar isometries we need to consider. There are four mirror reflections that map S_4 to itself:

$$\begin{array}{rcl}
m_1 = m_v : & (x,y) \mapsto (-x,y) & \text{with respect to the } y\text{-axis} \\
m_2 : & (x,y) \mapsto (y,x) & \text{with respect to the line } y = x \\
m_3 = m_h : & (x,y) \mapsto (x,-y) & \text{with respect to the } x\text{-axis} \\
m_4 : & (x,y) \mapsto (-y,-x) & \text{with respect to the line } y = -x
\end{array}$$

Note that

$$m_i(m_i(x,y)) = 1(x,y), \; i = 1,2,3,4.$$

There are also three (counterclockwise) rotations (about the origin 0=(0,0)):

$$\begin{array}{rc}
r_{\pi/2} : & (x,y) \mapsto (x\cos\pi/2 - y\sin\pi/2, x\sin\pi/2 + y\cos\pi/2) = (-y,x) \\
r_{\pi} : & (x,y) \mapsto (x\cos\pi - y\sin\pi, x\sin\pi + y\cos\pi) = (-x,-y) \\
r_{3\pi/2} : & (x,y) \mapsto (x\cos 3\pi/2 - y\sin 3\pi/2, x\sin 3\pi/2 + y\cos 3\pi/2) = (y,-x)
\end{array}$$

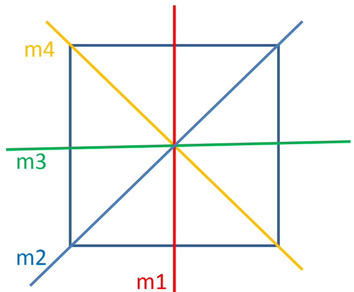
and $r_{2\pi} = 1$. Rotations are easy to combine among each others! For example

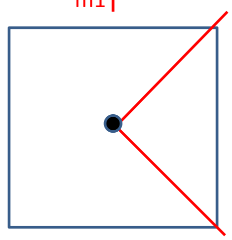
$$\begin{aligned}
r_{\pi} &= r_{\pi/2} r_{\pi/2} \\
r_{3\pi/2} &= r_{\pi/2} r_{\pi/2} r_{\pi/2}
\end{aligned}$$

and we can give the part of the multiplication table which involves only rotations. We summarize all the rotations by picking one rotation r whose powers contain the 4 rotations $r_{\pi/2}, r_{\pi}, r_{3\pi/2}, 1$. We can choose $r = r_{\pi/2}$ and $r = r_{3\pi/2}$, though in what follows we will focus on $r = r_{3\pi/2} = r_{-\pi/2}$, the rotation of 90 degrees clockwise, or 270 degrees counterclockwise:

	1	r	r^2	r^3
1	1	r	r^2	r^3
r	r	r^2	r^3	1
r^2	r^2	r^3	1	r
r^3	r^3	1	r	r^2

Symmetries of the Square (I)





What are the symmetries of the square?

There is the trivial symmetry 1.

There are mirror reflections:

1. Reflection in mirror m1
2. Reflection in mirror m2
3. Reflection in mirror m3
4. Reflection in mirror m4

There are rotations:

1. Rotation of 90 degrees
2. Rotation of 180 degrees
3. Rotation of 270 degrees

Symmetries of the Square (II)

- Let r = rotation of 90 degrees (clockwise), 270 degrees (counterclockwise)
- Let m denote the horizontal mirror reflection ($m=m_3$).
- Let 1 be the identity map.

Let us first look at rotations:
r^2 = rotation of 180 degrees
r^3= rotation of 270 degrees
r^4= rotation of 360 degrees = 1.

We now look at the mirror reflection m:
m^2=1.
(this is true for every mirror reflection!)

Let us try to compose mirror reflections with rotations. For that, we pick first

$$m = m_h : (x, y) \mapsto (x, -y),\ r = r_{3\pi/2} : (x, y) \mapsto (y, -x),$$

and compute what is rm and mr (you can choose to do the computations with another reflection instead of m_h, or with $r = r_{\pi/2}$ instead of $r = r_{3\pi/2}$.) We get

$$r(m(x, y)) = r(x, -y) = (-y, -x),\ m(r(x, y)) = m(y, -x) = (y, x)$$

and since $S_4 = \{(a, a), (-a, a), (a, -a)(-a, -a)\}$, we see that for example

$$r(m(a, a)) = (-a, -a),\ m(r(a, a)) = (a, a)$$

and these two transformations are different! We also notice something else which is interesting:

$$rm = m_4 = \text{reflection with respect to the line } y = -x$$

and

$$mr = m_2 = \text{reflection with respect to the line } y = x.$$

Since $rm \neq mr$ and we want to classify all the symmetries of the square S_4, we need to fix an ordering to write the symmetries in a systematic manner. We choose to first write a mirror reflection, and second a rotation (you could choose to first write a rotation and second a mirror reflection, what matters is that both ways allow you to describe all the symmetries, as we will see now!) This implies that we will look at all the possible following symmetries, written in the chosen ordering:

$$rm,\ r^2m,\ r^3m.$$

We have just computed rm, so next we have

$$r^2m(x, y) = r^2(x, -y) = r(-y, -x) = (-x, y) \tag{2.4}$$

and by applying r once more on (2.4) we get

$$r^3m(x, y) = r(-x, y) = (y, x)$$

showing that

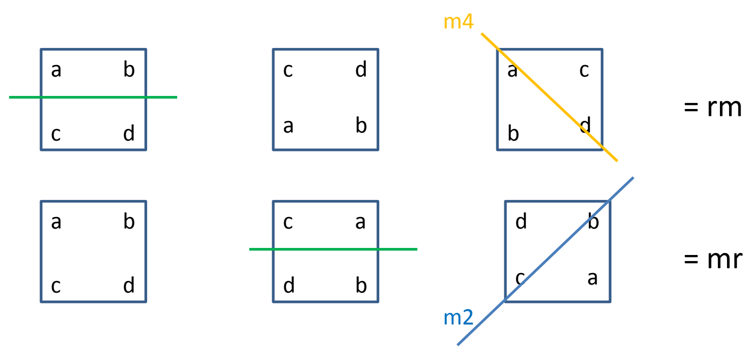
$$r^2m = \text{reflection with respect to the } y{-}axis$$

and

$$r^3m = mr = \text{reflection with respect to the line } y = x.$$

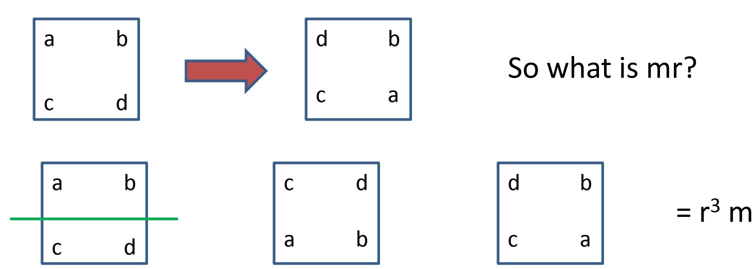
Symmetries of the Square (III)

- The composition of two symmetries = another symmetry!
- r=rotation of 90 deg (CW) or 270 deg (CCW), m=horizontal reflection



Symmetries of the Square (IV)

- We saw that mr is not equal to rm.
- Thus we need to decide an ordering to write the symmetries.
- We choose rm,r²m,r³m.



It is a good time to start summarizing all what we have been doing!

Step 1. We recognize that among all the planar isometries, there are 8 of them that are symmetries of the square S_4, namely:

1. m_1= reflection with respect to the y-axis,
2. m_2= reflection with respect to the line $y = x$,
3. m_3= reflection with respect to the x-axis,
4. m_4= reflection with respect to the line $y = -x$,
5. the rotation $r_{\pi/2}$,
6. the rotation r_π,
7. the rotation $r_{3\pi/2}$,
8. and of course the identity map 1!

Step 2. We fixed $m = m_3$ and $r = r_{3\pi/2}$ and computed all the combinations of the form $r^i m^j$, $i = 1, 2, 3, 4$, $j = 1, 2$, and we found that

$$\begin{aligned} rm &= m_4 \\ r^2m &= m_1 \\ r^3m &= m_2 \end{aligned}$$

which means that we can express all the above 8 symmetries of the square as $r^i m^j$, and furthermore, combining them does not give new symmetries!

We can thus summarize all the computations in the following multiplication table.

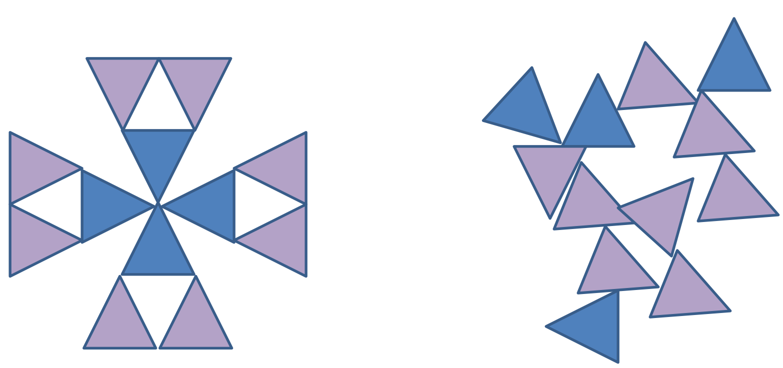
	1	m	r	r^2	r^3	rm	r^2m	r^3m
1	1	m	r	r^2	r^3	rm	r^2m	r^3m
m	m	1	r^3m	r^2m	rm	r^3	r^2	r
r	r	rm	r^2	r^3	1	r^2m	r^3m	m
r^2	r^2	r^2m	r^3	1	r	r^3m	m	rm
r^3	r^3	r^3m	1	r	r^2	m	rm	r^2m
rm	rm	r	m	r^3m	r^2m	1	r^3	r^2
r^2m	r^2m	r^2	rm	m	r^3m	r	1	r^3
r^3m	r^3m	r^3	r^2m	rm	m	r^2	r	1

Symmetries of the Square (V)

	1	m	r	r^2	r^3	rm	r^2m	r^3m
1	1	m	r	r^2	r^3	rm	r^2m	r^3m
m	m	1	r^3m	r^2m	rm	r^3	r^2	r
r	r	rm	r^2	r^3	1	r^2m	r^3m	m
r^2	r^2	r^2m	r^3	1	r	r^3m	m	rm
r^3	r^3	r^3m	1	r	r^2	m	rm	r^2m
rm	rm	r	m	r^3m	r^2m	1	r^3	r^2
r^2m	r^2m	r^2	rm	m	r^3m	r	1	r^3
r^3m	r^3m	r^3	r^2m	rm	m	r^2	r	1

Symmetries and Structure

A figure with many symmetries looks more structured!



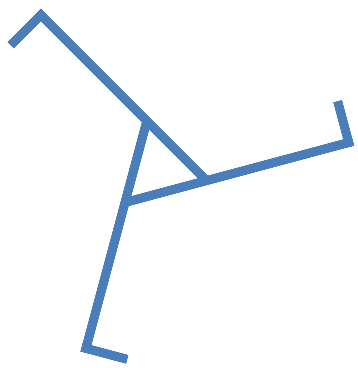
In the first chapter, we defined and classified planar isometries. Once we know what are all the possible isometries of plane, in this chapter, we focus on a subset of them: given a set of points S, what is the subset of planar isometries that preserves S. We computed three examples: (1) the symmetries of two points, (2) the symmetries of the rectangle, and (3) that of the square. We observed that the square has more symmetries (8 of them!) than the rectangle (4 of them). In fact, the more "regular" the set of points is, the more symmetries it has, and somehow, the "nicer" this set of points look to us!

Exercises for Chapter 2

Exercise 4. Determine the symmetries of an isosceles triangle, and compute the multiplication table of all its symmetries.

Exercise 5. Determine the symmetries of an equilateral triangle, and compute the multiplication table of all its symmetries.

Exercise 6. Determine the symmetries of the following shape, and compute the multiplication table of all its symmetries.



Exercise 7. Let $z = e^{2i\pi/3}$.

1. Show that $z^3 = 1$.
2. Compute the multiplication table of the set $\{1, z, z^2\}$.
3. Compare your multiplication table with that of Exercise 6. What can you observe? How would you interpret what you can see?