Chapter 6

Back to Geometry

"The noblest pleasure is the joy of understanding." (Leonardo da Vinci)

At the beginning of these lectures, we studied planar isometries, and symmetries. We then learnt the notion of group, and realized that planar isometries and symmetries have a group structure. After seeing several other examples of groups, such as integers mod n, and roots of unity, we saw through the notion of group isomorphism that most of the groups we have seen are in fact cyclic groups. In fact, after studying Lagrange Theorem, we discovered that groups of prime order are always cyclic, and the only examples of finite groups we have seen so far which are not cyclic are the Klein group (the symmetry group of the rectangle) and the symmetry group of the square. We may define the symmetry group of a regular polygon more generally.

Definition 14. The group of symmetries of a regular n-gon is called the Dihedral group, denoted by D_n.

In the literature, both the notation D_{2n} and D_n are found. We use D_n, where n refers to the number of sides of the regular polygon we consider.

Example 24. If $n = 3$, D_3 is the symmetry group of the equilateral triangle, while for $n = 4$, D_4 is the symmetry group of the square.

Recall so far

- We studied planar isometries.
- We extracted the notion of groups.
- We saw several examples of groups: integer mod n, roots of unity,…
- But after defining group isomorphism, we saw that many of them were just the same group *in disguise*: the cyclic group.

Cyclic groups are nice, but haven't we seen some other groups?

The Dihedral Group D_n

For n >2, the **dihedral group** is defined as the rigid motions of the plane preserving a regular n-gon, with respect to composition.

We saw

- D_3= group of symmetries of the equilateral triangle
- D_4= group of symmetries of the square

(In the literature, the notation D_n and D_{2n} are equally used.)

Recall that the group of symmetries of a regular polygon with n sides contains the n rotations $\{r_\theta,\ \theta = 2\pi k/n,\ k = 0,\dots,n-1\} = \langle r_{2\pi/n}\rangle$, together with some mirror reflections. We center this regular n-sided polygon at $(0,0)$ with one vertex at $(1,0)$ (we might scale it if necessary) and label its vertices by the nth roots of unity: $1,\omega,\omega^2,\dots,\omega^{n-1}$, where $\omega = e^{i2\pi/n}$. Now all its rotations can be written in the generic form of planar isometries $H(z) = \alpha z + \beta$, $|\alpha| = 1$ as

$$H(z) = \alpha z,\ \alpha = \omega^k = e^{i2\pi k/n},\ k = 0,\dots,n-1.$$

We now consider mirror reflections about a line l passing through $(0,0)$ at an angle φ_0, defined by $l(\lambda) = \lambda e^{i\varphi_0}, \lambda \in (-\infty,+\infty)$. To reflect a complex number $z = \rho e^{i\varphi}$ about the line l, let us write $z_R = \rho_R e^{i\varphi_R}$ for the complex number z after being reflected. Since a reflection is an isometry, $\rho_R = \rho$. To compute φ_R, suppose first that $\varphi_R \le \varphi_0$. Then $\varphi_R = \varphi + 2(\varphi_0 - \varphi)$. Similarly if $\varphi_R \ge \varphi_0$, $\varphi_R = \varphi - 2(\varphi - \varphi_0)$, showing that in both cases $\varphi_R = 2\varphi_0 - \varphi$. Hence

$$z_R = \rho e^{i\varphi_R} = \rho e^{i2\varphi_0 - i\varphi} = e^{i2\varphi_0}\rho e^{-i\varphi} = e^{i2\varphi_0}\overline{z}.$$

We now consider not any arbitrary complex number z, but when z is a root of unity ω^k. Mirror reflections that leave $\{1,\omega,\omega^2,\dots,\omega^{n-1}\}$ invariant, that is which map a root of unity to another, will be of the form

$$H(\omega^t) = e^{i\theta}\omega^{-t} = \omega^k$$

where $\theta = 2\varphi_0$ depends on the reflection line chosen. Then $e^{i\theta} = \omega^{k+t} = \omega^{(k+t)_{\text{mod n}}} = \omega^s$, and we find the planar isometries

$$H(z) = \omega^s\overline{z},\ s = 0,1,\dots,n-1.$$

Hence, given a vertex w^t, there are exactly two maps that will send it to a given vertex w^k: one rotation, and one mirror reflection. This shows that the order of D_n is $2n$.

Furthermore, defining a rotation r and a mirror reflection m by

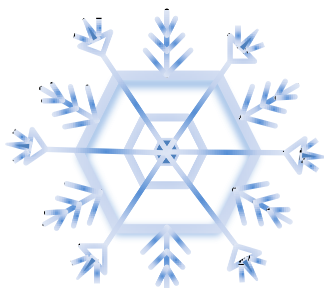
$$r : z \mapsto e^{i2\pi/n}z = \omega z,\ m : z \mapsto \overline{z}$$

we can write all the symmetries of a regular n-gon as

$$D_n = \{r^0 = 1, r, r^2, \dots, r^{n-1}, m, rm, r^m, \dots, r^{n-1}m\}.$$

In particular, $\omega^s\overline{z} = r^s m(z)$.

The Dihedral group D_6



Order of D_n

- We know: isometries of the plane are given by $z \to \alpha z + \beta$ and $z \to \alpha \bar{z} + \beta$, $|\alpha|=1$.
- Thus an element of D_n is either $z \to \alpha z$, or $z \to \alpha \bar{z}$.
- We may write the n vertices of a regular n-gon as nth roots of unity: $1, w, \ldots, w^{n-1}$.
- Now there are exactly 2 maps that send the vertex 1 to say the vertex w^k: $z \to w^k z$, and $z \to w^k \bar{z}$.

Thus the order of D_n is 2n.

These symmetries obey the following rules:

- $r^n = 1$, that is r is of order n, and $\langle r\rangle$ is a cyclic group of order n,
- $m^2 = 1$, that is m is of order 2, as $\overline{\overline{z}} = z$,
- $r^s m$ is also of order 2, as $(r^s m)(r^s m)(z) = \omega^s \overline{\omega^s \overline{z}} = \omega^s \omega^{-s} z = z$.

Since m and $r^s m$ are reflections, they are naturally of order 2, since repeating a reflection twice gives the identity map. Now

$$r^s m r^s m = 1 \Rightarrow m r^s m = r^{-s},\ \forall s \in \{0, 1, \ldots, n-1\}.$$

The properties

$$r^n = 1, m^2 = 1, mrm = r^{-1}$$

enable us to build the Cayley table of D_n. Indeed $\forall s, t \in \{0, 1, \ldots, n-1\}$

$$r^t r^s = r^{t+s \mod n},\ r^t r^s m = r^{t+s} m = r^{t+s \mod n} m,$$

and

$$m r^s = r^{-s} m = r^{n-s} m,\ r^t m r^s m = r^t r^{-s} = r^{t-s \mod n},\ r^t m r^s = r^t r^{-s} m = r^{t-s \mod n} m.$$

We see that D_n is not an Abelian group, since $r^s m \neq m r^s$. Hence we shall write

$$D_n = \{\langle r, m\rangle | m^2 = 1, r^n = 1, mr = r^{-1} m\},$$

that is, the group D_n is generated by r, m via concatenations of r's and m's reduced by the rules $r^n = 1, m^2 = 1, mrm = r^{-1}$ or $mr = r^{-1} m$.

Proof. Consider any string of r's and m's

$$\begin{aligned} &\underbrace{rr\cdot r}_{s_1}\underbrace{mm\cdots m}_{t_1}\underbrace{rr\cdots r}_{s_2}\underbrace{mm\cdots m}_{t_2}\cdots \\ =&r^{s_1} m^{t_1} r^{s_2} m^{t_2} r^{s_3} m^{t_3} \cdots r^{s_k} m^{t_k}. \end{aligned}$$

Due to $m^2 = 1$ and $r^n = 1$ we shall reduce this immediately to a string of

$$r^{\alpha_1} m r^{\alpha_2} m \cdots r^{\alpha_k} m$$

where $\alpha_i \in \{0, 1, \cdots, n-1\}$. Now using $m r^s m = r^{-s}$ gradually reduce all such strings, then we are done. $\square$

The Dihedral Group D_8

Description of the Dihedral Group

- The rotation $r:z \to wz$ generates a cyclic group $<r>$ of order n.
- The reflection $m: z \to \bar{z}$ is in the dihedral group but not in $<r>$.
- Thus $D_n = <r> \cup <r>m$.
- Furthermore: $mrm^{-1}(z) = mrm(z) = mr(\bar{z}) = m(w\bar{z}) = \bar{w}z = w^{-1}z = r^{-1}(z)$

$m^2=1$ $\quad m(z)=\bar{z}$ $\quad r(z)=wz$ $\quad$ w root of 1

That is $mrm^{-1} = r^{-1}$

This shows that: $D_n = \{ <r,m> \mid m^2=1, r^n=1, mr = r^{-1}m \}$

Indeed: we know we get 2n terms with $<r>$ and $<r>m$, and any term of the form mr^i can be reduced to an element in $<r>$ or $<r>m$ using $mr = r^{-1}m$: $mr^i = (mr)r^{i-1} = r^{-1}mr^{i-1} = r^{-1}(mr)r^{i-2}$ etc

What happens if $n = 1$ and $n = 2$? If $n = 1$, we have $r^1 = 1$, i.e., the group D_1 will be $D_1 = \{1, m\}$ with $m^2 = 1$, with Cayley table

	1	m
1	1	m
m	m	1

This is the symmetry group of a segment, with only one reflection or one 180° rotation symmetry.

If $n = 2$ we get $D_2 = \{1, r, m, rm\}$, with Cayley table

	1	r	m	rm
1	1	r	m	rm
r	r	1	rm	m
m	m	rm	1	r
rm	rm	m	r	1

This is the symmetry group of the rectangle, also called the Klein group.

Let us now look back.

- Planar isometries gave us several examples of finite groups:
  1. cyclic groups (rotations of a shape form a cyclic group)
  2. dihedral groups (symmetry group of a regular n-gon)
- Let us remember all the finite groups we have seen so far (up to isomorphism): cyclic groups, the Klein group, dihedral groups.

These observations address two natural questions:

Question 1. Can planar isometries give us other finite groups (up to isomorphism, than cyclic and dihedral groups)?

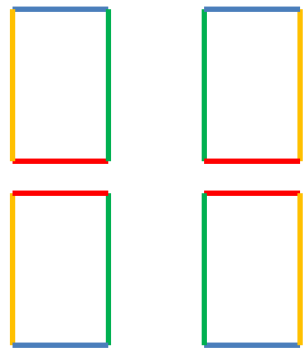
Question 2. Are there finite groups which are not isomorphic to subgroups of planar isometries?

We start with the first question, and study what are all the possible groups that appear as subgroups of planar isometries.

The Klein Group

When n=2, the description of D_2 gives the group of symmetries of the rectangle, also called the Klein group.

Christian Felix Klein (1849 –1925)



Two Natural Questions

Planar isometries gave us cyclic and dihedral groups. All our finite group examples so far are either cyclic or dihedral up to isomorphism.

QUESTION 1: can planar isometries give us other finite groups?

QUESTION 2: are there finite groups which are not isomorphic to planar isometries?

For that, let us recall what we learnt about planar isometries.

From Theorem 1, we know that every isometry in $\mathbb{R}^2$ can be written as $H : \mathbb{C} \to \mathbb{C}$, with

$$H(z) = \alpha z + \beta, \text{ or } H(z) = \alpha \bar{z} + \beta, \ |\alpha| = 1.$$

We also studied fixed points of planar isometries in Exercise 5. If $H(z) = \alpha z + \beta$, then

- if $\alpha = 1$, then $H(z) = z + \beta = z$ and there is no fixed point (apart if $\beta = 0$ and we have the identity map), and this isometry is a translation.
- if $\alpha \neq 1$, then $\alpha z + \beta = z \Rightarrow z = \frac{\beta}{1-\alpha}$, and

  $$H(z) - \frac{\beta}{1-\alpha} = \alpha z + \left(\beta - \frac{\beta}{1-\alpha}\right) = \alpha\left(z - \frac{\beta}{1-\alpha}\right)$$

  showing that $H(z) = \alpha\left(z - \frac{\beta}{1-\alpha}\right) + \frac{\beta}{1-\alpha}$, that is we translate the fixed point to the origin, rotate, and translate back, that is, we have a rotation around the fixed point $\frac{\beta}{1-\alpha}$.

If $H(z) = \alpha\bar{z} + \beta$, we first write this isometry in matrix form as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} \tag{6.1}$$

and fixed points (x_F, y_F) of this isometry satisfy the equation

$$\begin{bmatrix} x_F \\ y_F \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} x_F \\ y_F \end{bmatrix} + \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} \iff \underbrace{\begin{bmatrix} 1-\cos\theta & -\sin\theta \\ -\sin\theta & 1+\cos\theta \end{bmatrix}}_{M} \begin{bmatrix} x_F \\ y_F \end{bmatrix} = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

The matrix M has determinant $\det(M) = (1 - cos\theta)(1 + cos\theta) - sin^2\theta = 0$. By rewriting the matrix M as

$$M = \begin{bmatrix} 2sin\frac{\theta}{2}sin\frac{\theta}{2} & -2sin\frac{\theta}{2}cos\frac{\theta}{2} \\ -2sin\frac{\theta}{2}cos\frac{\theta}{2} & 2cos\frac{\theta}{2}cos\frac{\theta}{2} \end{bmatrix} = 2\begin{bmatrix} sin\frac{\theta}{2} \\ -cos\frac{\theta}{2} \end{bmatrix} \begin{bmatrix} sin\frac{\theta}{2} & -cos\frac{\theta}{2} \end{bmatrix}$$

and fixed points (x_F, y_F) have to be solutions of

$$2\begin{bmatrix} sin\frac{\theta}{2} \\ -cos\frac{\theta}{2} \end{bmatrix} \begin{bmatrix} sin\frac{\theta}{2} & -cos\frac{\theta}{2} \end{bmatrix} \begin{bmatrix} x_F \\ y_F \end{bmatrix} = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}.$$

First Question: Planar isometries

- Let us assume that we are given **a finite group** of planar isometries.
- What are all the isometries that could be in this finite group?

Remember all the isometries of the plane we saw in the first chapter?

- translations
- rotations
- reflection
- glide reflection = composition of reflection and translation

If $[t_1, t_2] = \lambda[\sin(\theta/2), -\cos(\theta/2)]$ then

$$2\langle[x_F, y_F], [\sin(\theta/2), -\cos(\theta/2)]\rangle = \lambda \Rightarrow x_F \sin(\theta/2) - y_F \cos(\theta/2) = \lambda/2$$

showing that (x_F, y_F) form a line, and the isometry (6.1) is now of the form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} & 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} & -\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \lambda \begin{bmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \end{bmatrix}$$
$$= \begin{bmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \lambda \begin{bmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \end{bmatrix}$$

Multiplying both sides by the matrix (rotation): $\begin{bmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \end{bmatrix}$ we get

$$\underbrace{\begin{bmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}}_{\begin{bmatrix} \tilde{x}' \\ \tilde{y}' \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \underbrace{\begin{bmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}}_{\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}} + \lambda \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and in the rotated coordinates $(\tilde{x}', \tilde{y}')$ and $(\tilde{x}, \tilde{y})$, we have $\tilde{x}' = \tilde{x}$ and $(\tilde{y}' - \frac{\lambda}{2}) = -(\tilde{y} - \frac{\lambda}{2})$ which shows that in the rotated coordinates this isometry is simply a reflection about the line $y = +\frac{\lambda}{2}$.

If $[t_1, t_2] \neq \lambda[\sin(\theta/2), -\cos(\theta/2)]$, then we have no fixed points. Just like in the previous analysis we have here

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} cos\frac{\theta}{2} & sin\frac{\theta}{2} \\ sin\frac{\theta}{2} & -cos\frac{\theta}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

and we have as before in the rotated coordinates that

$$\begin{bmatrix} \tilde{x}' \\ \tilde{y}' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} + \begin{bmatrix} cos\frac{\theta}{2} & sin\frac{\theta}{2} \\ sin\frac{\theta}{2} & -cos\frac{\theta}{2} \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} \quad = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} + \begin{bmatrix} m \\ n \end{bmatrix}$$

and we recognize a translation along the direction of the reflection line $\tilde{x}' = \tilde{x} + m$ and a reflection about the line $y = \frac{n}{2}$, since $(\tilde{y}' - \frac{n}{2}) = -(\tilde{y} - \frac{n}{2})$. This gives a proof of Theorem 2, which we recall here.

Planar Isometries in a Finite Group

- A translation generates an infinite subgroup!
- Thus translations cannot belong to a finite group.

• A glide reflection is the composition of a reflection and a translation.
• Thus again, it generates an infinite subgroup, and cannot belong to a finite group.

We are left with rotations and reflections!

Theorem 12. *Any planar isometry is either*

a) *A rotation about a point in the plane*

b) *A pure translation*

c) *A reflection about a line in the plane*

d) *A reflection about a line in the plane and a translation along the same line (glide reflection)*

Since we are interesting in subgroups of planar isometries, we now need to understand what happens when we compose isometries, since a a finite subgroup of isometries must be closed under composition.

A translation $T(\beta)$ is given by $T(\beta) : z \to z + \beta$, thus

$$T(\beta_2) \circ T(\beta_1) = (z + \beta_1) + \beta_2 = z + \beta_1 + \beta_2 = T(\beta_1 + \beta_2)$$

and translations form a subgroup of the planar isometries that is isomorphic to $(\mathbb{C}, +)$ or $(\mathbb{R}^2, +)$. The isomorphism f is given by $f : T(\beta) \mapsto \beta$.

A rotation R_Ω about a center $\Omega = z_0$ is given by

$$R_\Omega(\theta)z \to e^{i\theta}(z - z_0) + z_0,$$

thus

$$R_\Omega(\theta_2) \circ R_\Omega(\theta_1) = e^{i\theta_2}(e^{i\theta_1}(z - z_0) + z_0 - z_0) + z_0 = R_\Omega(\theta_1 + \theta_2)$$

which shows that rotations about a given fixed center $\Omega(= z_0)$ form a subgroup of the group of planar isometries.

We consider now the composition of two rotations about different centers:

$$R_{\Omega_1}(\theta_1) = e^{i\theta_1}(z - z_1) + z_1, \; R_{\Omega_2}(\theta_2) = e^{i\theta_2}(z - z_2) + z_2$$

so that

$$\begin{aligned} R_{\Omega_2}(\theta_2) \circ R_{\Omega_1}(\theta_1) &= e^{i\theta_2}(e^{i\theta_1}(z - z_1) + z_1 - z_2) + z_2 \\ &= e^{i(\theta_2+\theta_1)}(z - z_1) + e^{i\theta_2}(z_1 - z_2) + z_2 \\ &= e^{i(\theta_1+\theta_2)}[z - \gamma] + \gamma \end{aligned}$$

Rotations

- Recall: to define a rotation, we fix a center, say the origin, around which we rotate (counter-clockwise).

$$R(\theta) : z \rightarrow e^{\theta i} z$$

- What if we take a rotation around a point different than 0?

$$R_{z_0}(\theta) : z \rightarrow e^{\theta i} (z-z_0)+z_0$$

First translate z_0 to the origin, then rotate, then move back to z_0

Rotations around Different centers

- What if we take two rotations around different centers?

- If both $R_{z_1}(\theta_1)$ and $R_{z_2}(\theta_2)$ are in a finite group, then both their composition, and that of their inverse must be there!

$$R_{z_2}(\theta_2)\, R_{z_1}(\theta_1)(z)=e^{i(\theta_1+\theta_2)}z - e^{i(\theta_1+\theta_2)}z_1 + e^{i\theta_2}(z_1-z_2) +z_2$$
$$(R_{z_2}(\theta_2))^{-1}(R_{z_1}(\theta_1))^{-1}(z)= e^{-i(\theta_1+\theta_2)}z - e^{-i(\theta_1+\theta_2)}z_1 + e^{-i\theta_2}(z_1-z_2) +z_2$$
$$(R_{z_2}(\theta_2))^{-1}(R_{z_1}(\theta_1))^{-1}R_{z_2}(\theta_2)R_{z_1}(\theta_1)(z)=z+(z_2-z_1)[e^{-i(\theta_1+\theta_2)}-(e^{-i\theta_2}+ e^{-i\theta_1})+1]$$

Pure translation if z_1 is not z_2! Thus such rotations cannot be in a finite group!

where we determine γ:

$$\begin{aligned}
-e^{i(\theta_1+\theta_2)}z_1+e^{i\theta_2}z_1 - e^{i\theta_2}z_2 + z_2 &= -e^{i(\theta_1+\theta_2)}\gamma + \gamma \\
(1-e^{i(\theta_1+\theta_2)})\gamma &= z_2 + e^{i\theta_2}(z_1-z_2) - e^{i(\theta_1+\theta_2)}z_1 \\
\gamma &= \frac{z_2 + e^{i\theta_2}(z_1-z_2) - e^{i(\theta_1+\theta_2)}z_1}{1-e^{i(\theta_1+\theta_2)}}
\end{aligned}$$

Hence, we have a rotation by $(\theta_1+\theta_2)$ about a new center γ.

If $z_1 \neq z_2$ and $\theta_2 = -\theta_1$, we get in fact a translation:

$$\begin{aligned}
R_{\Omega_2}(-\theta_1)\circ R_{\Omega_1}(\theta_1) &= z - z_1 + e^{-i\theta_1}(z_1-z_2) + z_2 \\
&= z + \underbrace{(z_1-z_2)(e^{-i\theta_1}-1)}_{\text{a translation!}}
\end{aligned}$$

After rotations and translations, we are left with reflections and glide reflections about a line l. Suppose we have two reflections, or two glide reflections, of the form

$$\varphi_1 : z \to e^{i\theta_1}\bar{z} + \beta_1, \varphi_2 : z \to e^{i\theta_2}\bar{z} + \beta_2,$$

so that

$$\varphi_2\circ\varphi_1(z) = e^{i\theta_2}(\overline{e^{i\theta_1}\bar{z}+\beta_1}) + \beta_2 = e^{i(\theta_2-\theta_1)}z + \overline{\beta_1}e^{i\theta_2} + \beta_2.$$

Hence if $\theta_2 = \theta_1 = \theta$ we get a translation:

$$\varphi_2\circ\varphi_1(z) = z + \underbrace{\overline{\beta_1}e^{i\theta}+\beta_2}_{\text{a translation vector}}$$

which is happening when the lines defining the reflections and glide reflections are parallel (reflect a shape with respect to a line, and then again with respect to another line parallel to the first one, and you will see that the shape is translated in the direction perpendicular to the lines.)

If instead $\theta_2 - \theta_2 \neq 0$, we get a rotation, since the $\varphi_2\circ\varphi_1(z)$ will have one well defined **fixed point**, given by

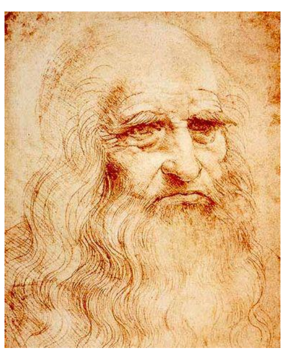
$$\begin{aligned}
z_{FP} &= e^{i(\theta_2-\theta_2)}z_{FP} + \overline{\beta_1}e^{i\theta_2} + \beta_2 \\
\Rightarrow z_{FP} &= \frac{\overline{\beta_1}e^{i\theta_2}+\beta_2}{1-e^{i(\theta_2-\theta_1)}}
\end{aligned}$$

Reflections

- Among the planar isometries, so far, **only rotations with same center z_0** are allowed!
- **Reflections** are also allowed, assuming that their lines intersect at z_0 (otherwise, we could get rotations about a different point.)

First Question: Leonardo Theorem

QUESTION 1: can planar isometries give us other finite groups than cyclic and dihedral groups?



ANSWER: No! This was already shown by Leonardo da Vinci!

Leonardo da Vinci (1452-1519) `` painter, sculptor, architect, musician, scientist, mathematician, engineer, inventor, anatomist, geologist, cartographer, botanist and writer " (dixit wikipedia)

Now, we have built up enough prerequisites to prove the following result.

Theorem 13 (Leonardo Da Vinci)**.** *The only finite subgroups of the group of planar symmetries are either C_n (the cyclic group of order n) or D_n (the dihedral group of order $2n$).*

Proof. Suppose that we have a finite subgroup $G = \{\varphi_1, \varphi_2, \cdots, \varphi_n\}$ of the group of planar isometries. This means that for every φ_k, $\langle\varphi_k\rangle$ is finite, that there exists $\varphi_k^{-1} \in G$, and that $\varphi_k \circ \varphi_l = \varphi_s \in G = \{\varphi_1, \varphi_2, \ldots, \varphi_n\}$. Thus

1. φ_k cannot be a translation, since $\langle\varphi_k\rangle = \{\varphi_k^n, n \in \mathbb{Z}\}$ is not a finite set.

2. φ_k cannot be a glide reflection, since $\varphi_k \circ \varphi_k$ is a translation hence $\langle\varphi_k^2\rangle$ is then not a finite set.

3. φ_k and φ_r cannot be rotations about different centers, since

$$R_{\Omega_2}(\theta_2)R_{\Omega_1}(\theta_1) = e^{i(\theta_2+\theta_1)}z - e^{i(\theta_2+\theta_1)}z_1 + e^{i\theta_2}(z_1 - z_2) + z_2$$
$$R_{\Omega_2}^{-1}(\theta_2)R_{\Omega_1}^{-1}(\theta_1) = e^{-i(\theta_2+\theta_1)}z - e^{-i(\theta_2+\theta_1)}z_1 + e^{-i\theta_2}(z_1 - z_2) + z_2$$

and

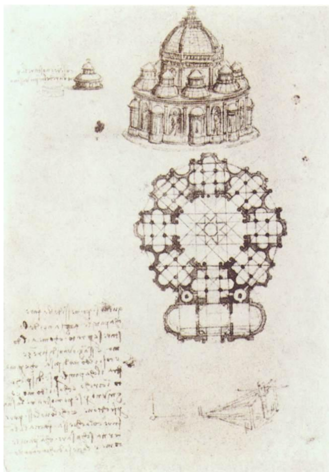
$$\begin{aligned}
&R_{\Omega_2}(-\theta_2)R_{\Omega_1}(-\theta_1)R_{\Omega_2}(\theta_2)R_{\Omega_1}(\theta_1)\\
&= e^{-i(\theta_2+\theta_1)}[e^{i(\theta_2+\theta_1)}z - e^{i(\theta_2+\theta_1)}z_1 + e^{i\theta_2}(z_1 - z_2) + z_2]\\
&\quad - e^{-i(\theta_2+\theta_1)}z_1 + e^{-i\theta_2}(z_1 - z_2) + z_2\\
&= z - z_1 + e^{-i\theta_1}(z_1 - z_2) + e^{-i(\theta_2+\theta_1)}z_2 - e^{-i(\theta_2+\theta_1)}z_1 + e^{-i\theta_2}(z_1 - z_2) + z_2\\
&= z + (z_2 - z_1) + e^{-i(\theta_2+\theta_1)}(z_2 - z_1) - (z_2 - z_1)(e^{-i\theta_1} + e^{i\theta_2})\\
&= z + \underbrace{(z_2 - z_1)[e^{-i(\theta_2+\theta_1)} - (e^{-i\theta_2} + e^{-i\theta_1}) + 1]}_{\text{a pure translation if } z_1 \neq z_2}
\end{aligned}$$

Therefore in the subgroup $G = \{\varphi_1, \varphi_2, \cdots, \varphi_n\}$ of finitely many isometries, we can have

1) rotations (which must all have the same center Ω)

2) reflections (but their lines must intersect at Ω otherwise we would be able to produce rotations about a point different from Ω and hence produce translations contradicting the finiteness of the set.)

Motivation for Leonardo Theorem



Leonardo da Vinci systematically determined all **possible symmetries** of a central building, and how to attach chapels and niches without destroying its symmetries.

Extract of Leonardo's notebooks.

Proof of Leonardo Theorem (I)

- We have already shown that a finite group of planar isometries can contain only rotations around the same center, and reflections through lines also through that center.
- Among all the rotations, take the one with smallest strictly positive angle θ, which generates a **finite cyclic group of order say n**, and every rotation belongs to this cyclic group!
- [if θ' is another rotation angle, then it is bigger than θ, thus we can decompose this rotation between a rotation of angle (a multiple of) θ and a smaller angle, a contradiction] ← same argument as we did several times for cyclic groups!

Let us look at the rotations about Ω in the subgroup $G = \{\varphi_1, \varphi_2, \cdots, \varphi_n\}$ and list the rotation angles (taken in the interval $[0, 2\pi)$) in increasing order:$\theta_1 < \theta_2 < \cdots < \theta_{l-1}$. Now $r(\theta_1)$ is the smallest rotation, and $r(2\theta_1)$, $r(3\theta_1)$,. . .,$r(k\theta_1)$ for all $k \in \mathbb{Z}$ must be in the subgroup as well.

We shall prove that these must be all the rotations in G, i.e., there cannot be a θ_t which is not $k\theta_1 \mod 2\pi$ for some k. Assume for the sake of contradiction that $\theta_t \neq k\theta_1$. Then $\theta_t = s\theta_1 + \zeta$ where $0 < \zeta < \theta_1$, and

$$r(\theta_t)r(-s\theta_1) = r(\theta_t)r(\theta_1)^{-s} = r(\zeta)$$

but $r(\theta_t)r(\theta_1)^{-s}$ belongs to the group of rotations and thus it is a rotation of an angle that belongs to $\{\theta_1, \theta_2, \cdots, \theta_{l-1}\}$, with $\zeta < \theta_1$ contradicting the assumption that θ_1 is the minimal angle.

Also note that $\theta_1 = 2\pi/l$ since otherwise $l\theta_1 = 2\pi + \eta$ with $\eta < \theta_1$ and $r^l(\theta_1) = r(\eta)$ with $\eta < \theta_1$, again contradicting the minimality of θ_1.

Therefore we have exactly l rotations generated by $r(\theta_1)$ and $\langle r(\theta_1)\rangle$ is the cyclic group C_l of order l.

If $C_l = \langle r(\theta_1)\rangle$ exhausts all the elements of $G = \{\varphi_1, \varphi_2, \cdots, \varphi_n\}$, we are done. If not, there are reflections in G too. Let m be a reflection that belongs to $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$. If m and $\langle r(\theta_1)\rangle$ are both in G, then by closure

$$m, mr, mr^2, \ldots, mr^{p-1} \in G$$

and all these are (1) reflections since $mr^\alpha = r^\beta \Rightarrow m = r^{(\beta-\alpha)}$ and m would be a rotation, (2) distinct elements since $mr^\alpha = mr^\beta \Rightarrow r^\alpha = r^\beta$.

Can another reflection be in the group say $\tilde{m}$? If $\tilde{m} \neq mr^\alpha$, then $m\tilde{m}$ is by definition a rotation in G, that is $m\tilde{m} = r^\alpha$, since we have shown that all rotations of G are in $\langle r(\theta_1)\rangle$. Now this shows that

$$\tilde{m} = m^{-1}r^\alpha = mr^\alpha, \text{ and } (mr^\alpha)(mr^\alpha) = 1 \Rightarrow mr^\alpha m = r^{-\alpha}.$$

Since $m^2 = 1$ as for any reflection, we proved that

$$G = \{1, r, r^2, \ldots, r^{l-1}, m, mr, \ldots, mr^{l-1}\},\ m^2 = 1,\ r^l = 1, mr^\alpha m = r^{-\alpha}.$$

The group G is therefore recognized as the dihedral group

$$D_p = \{\langle r, m\rangle | m^2 = 1, r^l = 1, mr = r^{-1}m\}.$$

Therefore we proved that a finite group of planar symmetries is either cyclic of some order l or dihedral of order $2l$ for some $l \in \mathbb{N}$. $\square$

Proof of Leonardo Theorem (II)

- If the finite group of isometries contain only rotations, done!
- If not, we have reflections!
- Let r be the rotation of smallest angle θ and m be a reflection.
- Then m, mr, mr^2,…, mr^{n-1} are distinct reflections that belong to the group [if $mr^i=r^j$ then m would be a rotation too].
- No other reflection! [for every reflection m', then mm' is a rotation, that is $mm'=r^j$ for some j, and m' is in the list!]

We proved: the finite group of planar isometries is either a **cyclic group** made of rotations, or a group of the form $\{1, r, r^2,.., r^{n-1}, m, mr, …, mr^{n-1}\}$ with relations $m^2=1$, $r^n=1$ and $mr^j = r^{-j}m$, namely the **dihedral group**!

Classification so far

(What we saw, no claim that this is complete ☺, all the finite ones written here are planar isometries)

Order	abelian groups	non-abelian groups
1	{1}	x
2	C_2	x
3	C_3	
4	C_4, Klein group	
5	C_5	
6	C_6	D_3
7	C_7	
8	C_8	D_4
infinite	$\mathbb{R}$	

Let us look at our table of small groups, up to order 8.

order n	abelian	non-abelian
1	$C_1 \simeq \{1\}$	**x**
2	C_2	**x**
3	C_3	**x**
4	C_4, Klein group	**x**
5	C_5	**x**
6	C_6	D_3
7	C_7	**x**
8	C_8	D_4

Using Leonardo Theorem, we know that planar isometries only provide cyclic and dihedral groups, so if we want to find potential more groups to add in this table, we cannot rely on planar geometry anymore! This leads to the second question we addressed earlier this chapter:

Are there finite groups which are not isomorphic to subgroups of the group of planar isometries?

Classification so far

Invertible mod 2,3,4,5,6,7 are cyclic, invertible mod 8 are C_2xC_2
[done in Exercises for 5 and 8, same computation for others!]

Order	abelian groups	non-abelian groups
1	{1}	x
2	C_2	x
3	C_3	
4	$C_{4,}$ Klein group	
5	C_5	
6	C_6	D_3
7	C_7	
8	C_8	D_4
infinite	$\mathbb{R}$	

We are left with the second Question…

QUESTION 2: are there finite groups which are not isomorphic to planar isometries?

Exercises for Chapter 6

Exercise 34. Show that any planar isometry of $\mathbb{R}^2$ is a product of at most 3 reflections.

Exercise 35. Look at the pictures on the wiki (available on edventure), and find the symmetry group of the different images shown.