Chapter 7

Permutation Groups

“.” ()

We started the study of groups by considering planar isometries. In the previous chapter, we learnt that finite groups of planar isometries can only be cyclic or dihedral groups. Furthermore, all the groups we have seen so far are, up to isomorphisms, either cyclic or dihedral groups! It is thus natural to wonder whether there are finite groups out there which cannot be interpreted as isometries of the plane. To answer this question, we will study next permutations. Permutations are usually studied as combinatorial objects, we will see in this chapter that they have a natural group structure, and in fact, there is a deep connection between finite groups and permutations!

We know intuitively what is a permutation: we have some objects from a set, and we exchange their positions. However, to work more precisely, we need a formal definition of what is a permutation.

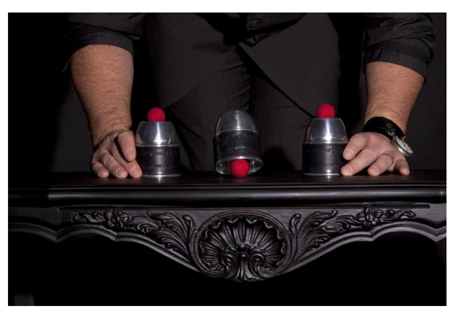
Question 2 after Lagrange Theorem

QUESTION 2: are there finite groups which are not isomorphic to planar isometries (cyclic or dihedral groups)?

Order	abelian groups	non-abelian groups
1	{1}	x
2	C_2	x
3	C_3	x
4	C_4, Klein group	x
5	C_5	x
6	C_6	D_3
7	C_7	x
8	C_8	D_4
infinite	$\mathbb{R}$	

What is a Permutation ? (I)

- Intuitively, we know what a permutation is…



http://www.virtualmagie.com/ubbthreads/ubbthreads.php/ubb/download/Number/3018/filename/3415%20net.jpg

Definition 15. A permutation of a set X is a function $\sigma : X \to X$ that is one-to-one and onto, i.e., a bijective map.

Let us make a small example to understand better the connection between the intuition and the formal definition.

Example 25. Consider a set X containing 3 objects, say a triangle, a circle and a square. A permutation of $X = \{\triangle, \circ, \square\}$ might send for example

$$\triangle \mapsto \triangle, \ \circ \mapsto \square, \ \square \mapsto \circ,$$

and we observe that what just did is exactly to define a bijection on the set X, namely a map $\sigma : X \to X$ defined as

$$\sigma(\triangle) = \triangle, \ \sigma(\circ) = \square, \ \sigma(\square) = \circ.$$

Since what matters for a permutation is how many objects we have and not the nature of the objects, we can always consider a permutation on a set of n objects where we label the objects by $\{1, \ldots, n\}$. The permutation of Example 25 can then be rewritten as $\sigma : \{1, 2, 3\} \to \{1, 2, 3\}$ such that

$$\sigma(1) = 1, \ \sigma(2) = 3, \ \sigma(3) = 2, \text{ or } \sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

Permutation maps, being bijective, have inverses and the maps combine naturally under composition of maps, which is associative. There is a natural identity permutation $\sigma : X \to X$, $X = \{1, 2, 3, \ldots, n\}$ which is

$$\sigma(k) \mapsto k.$$

Therefore all the permutations of a set $X = \{1, 2, \ldots, n\}$ form a group under composition. This group is called the symmetric group S_n of degree n.

What is the order of S_n? Let us count how many permutations of $\{1, 2, \ldots, n\}$ we have. We have to fill the boxes

			$\cdots$	
1	2	3	$\cdots$	n

with numbers $\{1, 2, \ldots, n\}$ with no repetitions. For box 1, we have n possible candidates. Once one number has been used, for box 2, we have $(n-1)$ candidates, ... Therefore we have

$$n(n-1)(n-2)\cdots 1 = n!$$

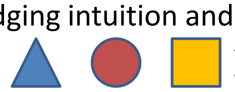
permutations and the order of S_n is

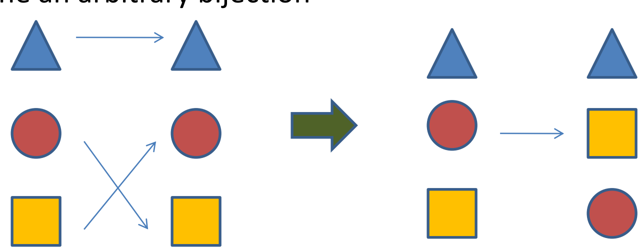
$$|S_n| = n!.$$

What is a Permutation? (II)

- What is formally a permutation?
- A **permutation** of an arbitrary set X is a bijection from X to itself
- Recall that a bijection is both an injection and a surjection.

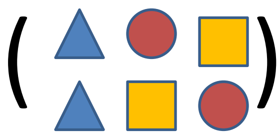
What is a Permutation? (III)

- Bridging intuition and formalism
- X={  }
- Define an arbitrary bijection



Notation

If |X|=n, we label the n elements by 1…n.



$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

Combining Permutations

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

> It's a composition, so this permutation first!

1 2 3 → 1 3 2 → 2 3 1

Group Structure of Permutations (I)

- All permutations of a set X of n elements form a group under composition, called the **symmetric group** on n elements, denoted by S_n.

Composition of two bijections is a bijection

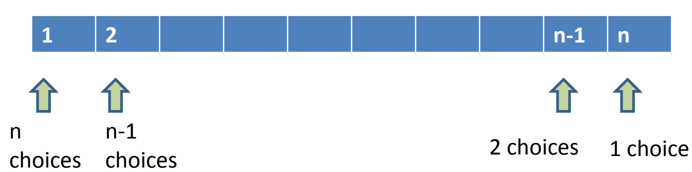
- Identity = do-nothing (do no permutation)
- Every permutation has an inverse, the inverse permutation.

A permutation is a bijection!

- Non abelian (the two permutations of the previous slide do not commute for example!)

Group Structure of Permutations (II)

The order of the group S_n of permutations on a set X of elements is n!



$|S_n| = n!$

Let us see a few examples of symmetric groups S_n.

Example 26. If $n = 1$, S_1 contains only one element, the permutation identity!

Example 27. If $n = 2$, then $X = \{1, 2\}$, and we have only two permutations:

$$\sigma_1 : 1 \mapsto 1,\ 2 \mapsto 2$$

and

$$\sigma_2 : 1 \mapsto 2,\ 2 \mapsto 1,$$

and $S_2 = \{\sigma_1, \sigma_2\}$. The Cayley table of S_2 is

	σ_1	σ_2
σ_1	σ_1	σ_2
σ_2	σ_2	σ_1

.

Let us introduce the *cycle notation*. We write (12) to mean that 1 is sent to 2, and 2 is sent to 1. With this notation, we write

$$S_2 = \{(), (12)\}.$$

This group is isomorphic to C_2, and it is abelian.

The permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

of Example 25 in the cycle notation is written as (23). We can combine two such permutations:

$$(12)(23)$$

which means that we first permute 2 and 3: $1\ 2\ 3 \mapsto 1\ 3\ 2$ and then we permute 1 and 2: $1\ 3\ 2 \mapsto 2\ 3\ 1$. Let us look next at the group S_3.

Permutations on a Set of 2 Elements

- $|X| = 2$, $X = \{ 1, 2\}$
- $|S_2| = 2$, $S_2 = \{ \sigma_1, \sigma_2\}$, $\sigma_1 : 1\ 2 \rightarrow 1\ 2$, $\sigma_2 : 1\ 2 \rightarrow 2\ 1$.

	σ_1	σ_2
σ_1	σ_1	σ_2
σ_2	σ_2	σ_1

Cycle Notation

$\begin{pmatrix} 1\ 2\ 3 \\ 1\ 3\ 2 \end{pmatrix}$ (23) ➡

2 → 3
3 → 2
thus 123 → 132

$\begin{pmatrix} 1\ 2\ 3 \\ 2\ 1\ 3 \end{pmatrix} \begin{pmatrix} 1\ 2\ 3 \\ 1\ 3\ 2 \end{pmatrix}$ (12)(23) ➡

2 → 3
3 → 2
thus 123 → 132
1 → 2
2 → 1
thus 132→231

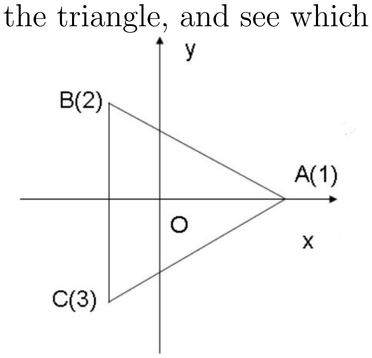
Example 28. If $n = 3$, we consider the set $X = \{1, 2, 3\}$. Since $3! = 6$, we have 6 permutations:

$$S_3 = \{\sigma_1 = (), \sigma_2 = (12), \sigma_3 = (13), \sigma_4 = (23), \sigma_5 = (123), \sigma_6 = (132)\}.$$

We compute the Cayley table of S_3.

	()	(12)	(23)	(13)	(123)	(132)
()	()	(12)	(23)	(13)	(123)	(132)
(12)	(12)	()	(123)	(132)	(23)	(13)
(23)	(23)	(132)	()	(123)	(13)	(12)
(13)	(13)	(123)	(132)	()	(12)	(23)
(123)	(123)	(13)	(12)	(23)	(132)	()
(132)	(132)	(23)	(13)	(12)	()	(123)

We see from the Cayley table that S_3 is indeed isomorphic to D_3! This can also be seen geometrically as follows. Consider an equilateral triangle, and label its 3 vertices by A, B, C, and label the locations of the plane where each is by 1,2,3 (thus vertex A is at location 1, vertex B at location 2 and vertex C as location 3). Let us now rotate the triangle by r (120 degrees counterclockwise), to find that now, at position 1 we have C, at position 2 we have A and at position 3 we have B, and we apply all the symmetries of the triangle, and see which vertex is sent to position 1,2, and 3 respectively:



	1	2	3	
1	A	B	C	()
r	C	A	B	(213)
r^2	B	C	A	(123)
m	A	C	B	(23)
rm	B	A	C	(12)
r^2m	C	B	A	(13)

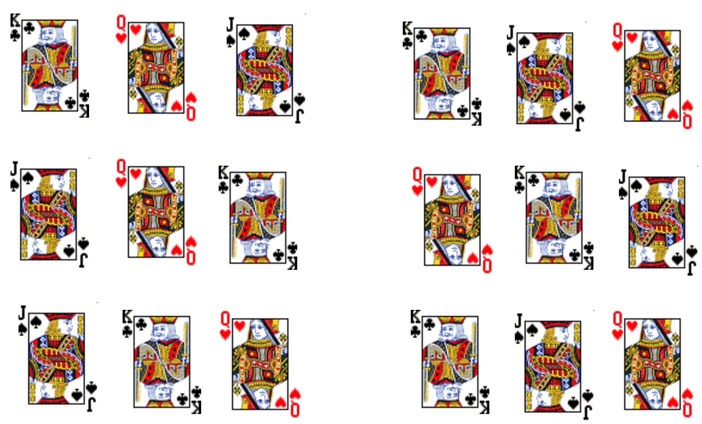
and we see that to each symmetry corresponds a permutation. For example, r sends ABC to CAB and thus we have (132).

Permutations on a Set of 3 Elements

- |X|=3, X={1 2 3}
- σ_1 :123 →123 (), σ_2 :123 →213 (12) ,σ_3 :123→321 (13) ,
  σ_4 :123 →132 (23) , σ_5: 123→231 (123) , σ_6 : 123→312 (132) .

	()	(1,2)	(2,3)	(1,3)	(1,2,3)	(1,3,2)
()	()	(1,2)	(2,3)	(1,3)	(1,2,3)	(1,3,2)
(1,2)	(1,2)	()	(1,2,3)	(1,3,2)	(2,3)	(1,3)
(2,3)	(2,3)	(1,3,2)	()	(1,2,3)	(1,3)	(1,2)
(1,3)	(1,3)	(1,2,3)	(1,3,2)	()	(1,2)	(2,3)
(1,2,3)	(1,2,3)	(1,3)	(1,2)	(2,3)	(1,3,2)	()
(1,3,2)	(1,3,2)	(2,3)	(1,3)	(1,2)	()	(1,2,3)

The Symmetric Group S_3



Have we found New Groups?

- S_2 ?

  since | S_2 |=2, it is the cyclic group C_2!
- S_3 ?

  We know $|S_3|=3!=6$, and it is non-abelian.
  We also know $|D_3|=2\cdot3=6$ and it is non-abelian.

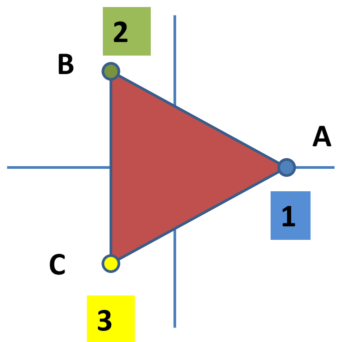
S_3 vs D_3

	()	(12)	(23)	(13)	(123)	(132)
()	()	(1,2)	(2,3)	(1,3)	(123)	(132)
(1,2)	(1,2)	()	(123)	(132)	(2,3)	(1,3)
(2,3)	(2,3)	(132)	()	(123)	(1,3)	(1,2)
(1,3)	(1,3)	(123)	(132)	()	(1,2)	(2,3)
(123)	(123)	(1,3)	(1,2)	(2,3)	(132)	()
(132)	(132)	(2,3)	(1,3)	(1,2)	()	(123)

	1	r	r^2	m	rm	r^2m
1	1	r	r^2	m	rm	r^2m
r	r	r^2	1	rm	r^2m	m
r^2	r^2	1	r	r^2m	m	rm
m	m	r^2m	rm	1	r^2	r
rm	rm	m	r^2m	r	1	r^2
r^2m	r^2m	rm	m	r^2	r	1

Are they isomorphic?

D_3 revisited



- Fix 3 locations on the plane: 1, 2, 3
- Call A,B,C the 3 triangle vertices

	1	2	3	
1	A	B	C	()
r	C	A	B	(213)
r^2	B	C	A	(123)
m	A	C	B	(23)
rm	B	A	C	(12)
r^2m	C	B	A	(13)

Question 2: more Bad News !

QUESTION 2: are there finite groups which are not isomorphic to planar isometries (cyclic or dihedral groups)?

Order	abelian groups	non-abelian groups
1	{1}	x
2	$C_2 = S_2$	x
3	C_3	x
4	C_4, Klein group	x
5	C_5	x
6	C_6	$D_3 = S_3$
7	C_7	x
8	C	D
infinite		

More work is needed!

Thus despite the introduction of a new type of groups, the groups of permutations, we still have not found a finite group which is not a cyclic or a dihedral group. We need more work! For that, we start by noting that permutations can be described in terms of matrices.

Any permutation σ of the elements $\{1, 2, \ldots, n\}$ can be described by

$$\begin{bmatrix} \sigma(1) \\ \sigma(2) \\ \vdots \\ \sigma(n) \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 1 & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 1 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ \vdots \\ n \end{bmatrix} = \underbrace{\begin{bmatrix} e^T_{\sigma(1)} \\ \vdots \\ e^T_{\sigma(n)} \end{bmatrix}}_{P_\sigma} \begin{bmatrix} 1 \\ \vdots \\ n \end{bmatrix},$$

where the kth row of the binary matrix is given by $e^T_{\sigma(k)} = (0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 is at location $\sigma(k)$. Now $e_1, \ldots, e_n$ are a set of orthogonal vectors, that is, satisfying

$$e_k^T e_s = \langle e_k, e_s \rangle = \delta_{ks} = \begin{cases} 0 & if\ k \neq s \\ 1 & if\ k = s \end{cases}, \tag{7.1}$$

which form a standard basis of $\mathbb{R}^n$. Let us derive some properties of the matrix P_σ.

Property 1. The matrix P_σ is orthogonal, that is $P_\sigma P_\sigma^T = I_n$, where I_n is the identity matrix. This follows from

$$\begin{bmatrix} e^T_{\sigma(1)} \\ e^T_{\sigma(2)} \\ \vdots \\ e^T_{\sigma(n)} \end{bmatrix} \begin{bmatrix} e^T_{\sigma(1)} & e^T_{\sigma(2)} & \cdots & e^T_{\sigma(n)} \end{bmatrix} = \begin{bmatrix} & \vdots & \\ \cdots & \langle e_{\sigma(i)}, e_{\sigma(j)} \rangle & \cdots \\ & \vdots & \end{bmatrix} = I_n$$

using (7.1). Hence the inverse of a permutation matrix is its transpose.

Property 2. Using that $\det(AB) = \det(A)\det(B)$ and $\det(A^T) = \det(A)$, we get

$$\det(P_\sigma P_\sigma^T) = \det(I) = 1.$$

Therefore $\det(P_\sigma) = \pm 1$. ($\det(P_\sigma^T) = \det(P_\sigma) \Rightarrow (\det P_\sigma)^2 = 1$).

Permutation Matrices : Definition

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

If σ is a permutation on X={1…n}, then it can be represented by a permutation matrix P_σ

kth row has a 1 at position σ(k) (0…0 1 0…0)

$$\begin{bmatrix} 1 & 0 & 0 \\ & \cdots & \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ \vdots \\ n \end{bmatrix} = \begin{bmatrix} \sigma(1) \\ \vdots \\ \sigma(n) \end{bmatrix}$$

Permutation Matrices: Properties

Every row/column has only a 1

$$P_\sigma P_\sigma^T = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}\begin{bmatrix} p_1^T & \cdots & p_n^T \end{bmatrix} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

A permutation matrix as an orthogonal matrix!

$\det(P_\sigma P_\sigma^T)=1 \Rightarrow \det(P_\sigma)= 1$ or -1

Property 3. We will show next that any permutation can be decomposed as a chain of "elementary" permutations called transpositions, or exchanges.

We consider the permutation σ given by

$$\begin{bmatrix} \sigma(1) \\ \vdots \\ \sigma(n) \end{bmatrix} = \begin{bmatrix} e_{\sigma(1)}^T \\ \vdots \\ e_{\sigma(n)}^T \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ n \end{bmatrix}.$$

We shall produce σ from $(1, 2, \ldots, n)$ by successively moving $\sigma(1)$ to the first place and 1 to the place of $\sigma(1)$, then $\sigma(2)$ to the second place and whoever is in the second place after the first exchange to the place of $\sigma(2)$ place, etc..

After moving $\sigma(1)$ to the first place, using a matrix P, we get

$$\begin{bmatrix} \sigma(1) \\ 2 \\ \vdots \\ 1 \\ \vdots \\ n \end{bmatrix} = P_{n\times n} \begin{bmatrix} 1 \\ 2 \\ \vdots \\ \sigma(1) \\ \vdots \\ n \end{bmatrix}.$$

After this step, we use an $(n-1)\times(n-1)$ permutation matrix to bring $\sigma(2)$ to the second place as follows (without affecting $\sigma(1)$):

$$\begin{bmatrix} \sigma(1) \\ \sigma(2) \\ \vdots \\ 2 \\ \vdots \\ n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & P_{(n-1)\times(n-1)} \end{bmatrix} \begin{bmatrix} \sigma(1) \\ 2 \\ \vdots \\ \sigma(2) \\ \vdots \\ n \end{bmatrix},$$

and so on. From this process, it is clear that at every stage we have either a matrix of exchange in which two rows of the identity are exchanged, or if the output happens to have the next value in its designated place an identity matrix. The process will necessarily terminate after n steps and will yield the permutation σ as desired.

Transpositions

A **transposition** (exchange) is a permutation that swaps two elements and does not change the others.

- In cycle notation, a transposition has the form (i j).

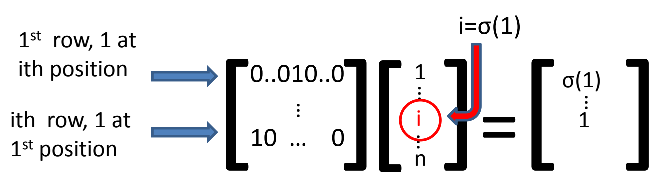
Example: (1 2) on the set X={1,2,3,4} means 1234 →2134.

- In matrix notation, a transposition is an identity matrix, but for for two rows that are swapped.

$$\begin{bmatrix} 0100 \\ 1000 \\ 0010 \\ 0001 \end{bmatrix}$$

Decomposition in Transpositions (I)

Any permutation can be decomposed as a product of transpositions.



Place similarly σ(2) at the 2nd position, σ(3) at the 3rd position etc, this process stops at most after n steps! (since at every step, either two rows are exchanged, or we have an identity matrix if nothing needs to be changed).

Hence we will be able to write

$$\begin{bmatrix} \sigma(1) \\ \sigma(2) \\ \vdots \\ \sigma(n) \end{bmatrix} = E_n E_{n-1} \cdots E_2 E_1 \begin{bmatrix} 1 \\ 2 \\ \vdots \\ n \end{bmatrix}$$

where $E_i = \begin{cases} \text{either an elementary exchange matrix of size n} \times \text{n} \\ \text{or an identity matrix of size n} \times \text{n} \end{cases}.$

Now, we know from the property of the determinant that exchanging two rows in a matrix induces a sign change in the determinant. Hence we have

$$\det E_i = \begin{cases} -1 & \text{if it is a proper exchange} \\ 1 & \text{if it is } I_{n\times n} \end{cases}.$$

Therefore we have shown that for any permutation, we have a decomposition into a sequence of transpositions (or exchanges), and we need at most n of them to obtain any permutation. Hence for any σ we have:

$$P_\sigma = E_n E_{n-1} \cdots E_1$$

and

$$\det P_\sigma = \det E_n \det E_{n-1} \cdots \det E_1 = (-1)^{\#\text{ of exchanges}}.$$

Example

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \rightarrow \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \quad \sigma(1)=3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \quad \sigma(2)=1$$

Decomposition in Transpositions (II)

$$\underbrace{E_n \ldots E_2 E_1}_{P_\sigma} \begin{bmatrix} 1 \\ n \end{bmatrix} = \begin{bmatrix} \sigma(1) \\ \sigma(n) \end{bmatrix}$$

where E_i is either an identity matrix, or a transposition (exchange) matrix.

$\det(E_i)=-1$ for a transposition, and 1 for the identity, thus **$\det(P_\sigma)=(-1)^{\#exchanges}$**

The above development enable us to define the permutation to be even if

$$\det P_\sigma = 1,$$

and odd if

$$\det P_\sigma = -1.$$

Definition 16. The sign/signature of a permutation σ is the determinant of P_σ. It is either 1 if the permutation is even or -1 otherwise.

We have a natural way to combine permutations as bijective maps. In matrix form, we have that if

$$P_{\sigma_A}\begin{bmatrix}1\\ \vdots\\ n\end{bmatrix}=\begin{bmatrix}\sigma_A(1)\\ \vdots\\ \sigma_A(n)\end{bmatrix}, P_{\sigma_B}\begin{bmatrix}1\\ \vdots\\ n\end{bmatrix}=\begin{bmatrix}\sigma_B(1)\\ \vdots\\ \sigma_B(n)\end{bmatrix}$$

then

$$P_{\sigma_A}P_{\sigma_B} = P_{\sigma_B\circ\sigma_A}.$$

The description of a permutation via transposition is not unique but the parity is an invariant. We also have that

$$\begin{aligned} sign(\sigma_A\circ\sigma_B) &= sign(\sigma_A)sign(\sigma_B)\\ \det(P_{\sigma_B}P_{\sigma_A}) &= \det(P_{\sigma_B})\det(P_{\sigma_A}). \end{aligned}$$

Then we have the multiplication rule.

	even	odd
even	even	odd
odd	odd	even

This shows the following.

Theorem 14. *All even permutations form a subgroup of permutations.*

Proof. Clearly the identity matrix is an even permutation, since its determinant is 1.

Product of even permutations is even, thus closure is satisfied.

The inverse of an even permutation must be even. To show this, we know

$$P_\sigma^T P_\sigma = I,$$

so $\det(P_\sigma^T) = \det(P_\sigma) \Rightarrow \det(P_\sigma^T) = 1$ if $\det(P_\sigma) = 1$. □

Definition 17. The subgroup A_n of even permutations of the symmetric group S_n is called the alternating group.

Parity of a Permutation

A permutation is **even** if $\det(P_\sigma)=1$ and **odd** if $\det(P_\sigma)=-1$.
The sign/signature of a permutation σ is $\text{sign}(\sigma)=\det(P_\sigma)$.

Example
(132) : 123 →312
123 → 312 thus (13) : 123 →321
321→312 thus (12)(13) : 123 →321→312

$\text{sign}(132)=(-1)^2=1$.

Same result from the matrix approach!

The decomposition in transpositions is far from unique! It is the signature which is unique!!

The Alternating Group

The subset of S_n formed by even permutations is a group, called the **alternating group** A_n.

- The identity is the do-nothing permutation σ= (), its permutation matrix is the identity, and its determinant is 1 and sign(())=1, that is () is even.
- The composition of two even permutations is even, since $\det(P_{\sigma1}P_{\sigma2})= \det(P_{\sigma1}) \det(P_{\sigma2})=1\cdot1=1$.
- If σ is a permutation with matrix P_σ, then its inverse permutation has matrix P_σ^T . Now $\det(P_\sigma P_\sigma^T)=1$ and since $\det(P_\sigma)=1$, we must have $\det(P_\sigma^T)=1$!

Example 29. When $n = 3$, we consider the symmetric group S_3, and identify those permutations which are even. Among the 6 permutations of S_3, 3 are odd and 3 are even. Thus A_3 is isomorphic to the cyclic group C_3 of order 3.

An interesting immediate fact is that the size of the subgroup of even permutations is $\frac{1}{2}n!$, since for every even permutation, one can uniquely associate an odd one by exchanging the first two elements!

Let us go back once more to our original question. We are looking for a group which is not isomorphic to a group of finite planar isometries. Since A_3 is isomorphic to a cyclic group, let us consider the next example, namely A_4.

Since $4! = 24$, we know that $|A_4| = 12$. There is a dihedral group D_6 which also has order 12. Are the two groups isomorphic?

Lagrange theorem tells us that elements of A_4 have an order which divides 12, so it could be 1,2,3,4 or 12. We can compute that there are exactly 3 elements of order 2:

$$(12)(34),\ (13)(24),\ (14)(23),$$

and 8 elements of order 3:

$$(123),\ (132),\ (124),\ (142),\ (134),\ (143),\ (234),\ (243).$$

This shows that A_4 and D_6 cannot be isomorphic! We thus just found our first example, to show that there is more than cyclic and dihedral groups!

Example: A_3

	()	(12)	(23)	(13)	(123)	(132)
()	()	(1,2)	(2,3)	(1,3)	(123)	(132)
(1,2)	(1,2)	()	(123)	(132)	(2,3)	(1,3)
(2,3)	(2,3)	(132)	()	(123)	(1,3)	(1,2)
(1,3)	(1,3)	(123)	(132)	()	(1,2)	(2,3)
(123)	(123)	(1,3)	(1,2)	(2,3)	(132)	()
(132)	(132)	(2,3)	(1,3)	(1,2)	()	(123)

	()	(123)	(132)
()	()	(123)	(132)
(123)	(123)	(132)	()
(132)	(132)	()	(123)

It is the cyclic group of order 3!

Order of A_n

The order of A_n is $|A_n|=|S_n|/2 = n!/2$.

Proof. To every even permutation can be associated uniquely an odd one by permuting the first two elements!

Examples.

- A_2 is of order 1 ⟹ this is {1}.
- A_3 is of order 3!/2=6/2=3 ⟹ this is C_3.
- A_4 is of order 4!/2 =24/2=12 ⟹ ?

Question 2: one more Bad News ??

QUESTION 2: are there finite groups which are not isomorphic to planar isometries (cyclic or dihedral groups)?

Order	abelian groups	non-abelian groups
1	{1}	x
2	**$C_2 = S_2$**	x
3	C_3	x
4	C_4, Klein group	x
5	C_5	x
6	C_6	**$D_3 = S_3$**
7	C_7	x
8	C_8	D_4
12	C_{12}	D_6, A_4

Order of Elements in A_4

- Lagrange Theorem tells us: 1,2,3,4,6,12.
- In fact: 3 elements of order 2, namely (12)(34), (13)(24), (14)(23)
- And 8 elements of order 3, namely (123),(132),(124),(142),(134),(143),(234),(243)

A_4 and D_6 are not isomorphic!

http://kristin-williams.blogspot.com/2009/09/yeah.html

Exercises for Chapter 7

Exercise 36. Let σ be a permutation on 5 elements given by $\sigma = (15243)$. Compute $\text{sign}(\sigma)$ (that is, the parity of the permutation).

Exercise 37.

1. Show that any permutation of the form (ijk) is always contained in the alternating group A_n, $n \geq 3$.
2. Deduce that A_n is a non-abelian group for $n \geq 4$.

Exercise 38. Let $H = \{\sigma \in S_5 \mid \sigma(1) = 1,\ \sigma(3) = 3\}$. Is H a subgroup of S_5?

Exercise 39. In the lecture, we gave the main steps to show that the group D_6 cannot be isomorphic to the group A_4, though both of them are of order 12 and non-abelian. This exercise is about filling some of the missing details.

- Check that $(1\ 2)(3\ 4)$ is indeed of order 2.
- Check that $(1\ 2\ 3)$ is indeed of order 3.
- By looking at the possible orders of elements of D_6, prove that A_4 and D_6 cannot be isomorphic.