Chapter 12

Solutions to the Exercises

"Intuition comes from experience, experience from failure, and failure from trying."

Exercises for Chapter 1

Exercise 1. Show that 2 is the only prime number which is even.

Solution. Take p a prime number. Then p has only 2 divisors, 1 and p. If p is also even, then one of its divisors has to be 2, thus $p = 2$.

Exercise 2. Show that if n^2 is even, then n is even, for n an integer.

Solution. An integer n is either even, that is $n = 2n'$, for some integer n', or odd, that is $n = 2n' + 1$ for some integer n'. Thus n^2 is either $4(n')^2$ or $4(n')^2 + 4n' + 1$. The case where n^2 is even is thus when $n = 2n'$.

Exercise 3. The goal of this exercise is to show that $\sqrt{2}$ is irrational. We provide a step by step way of doing so.

1. Suppose by contradiction that $\sqrt{2}$ is rational, that is $\sqrt{2} = \frac{m}{n}$, for m and n integers with no common factor. Show that m has to be even, that is $m = 2k$.

2. Compute m^2, and deduce that n has to be even too, a contradiction.

Solution. 1. Suppose by contradiction that $\sqrt{2}$ is rational, that is $\sqrt{2} = \frac{m}{n}$, for m and n integers with no common factor. Then

$$2 = \frac{m^2}{n^2}$$

and thus $m^2 = 2n^2$, showing that m^2 is even, that is, using Exercise 2, m has to be even, say $m = 2k$ for k some integer.

2. Now $m^2 = (2k)^2 = 4k^2$. This tells us, combining with the first step of the exercise, that

$$m^2 = 4k^2 = 2n^2$$

which implies that $2k^2 = n^2$, that is n^2 is even and by again by Exercise 2, it must be that n is even. This is a contradiction, since we assumed that m and n have no commun factor.

Exercise 4. This exercise is optional, it requires to write things quite formally. Show the following two properties of integers modulo n:

1. $(a \mod n) + (b \mod n) \equiv (a + b) \mod n$.
2. $(a \mod n)(b \mod n) \equiv (a \cdot b) \mod n$.

Solution. 1. Suppose $(a \mod n) = a'$, that is $a = qn+a'$, and $(b \mod n) = b'$, that is $b = rn + b'$, for some integers q, r. Then

$$(a \mod n) + (b \mod n) \equiv a' + b' \mod n$$

and

$$(a + b) \mod n \equiv (qn + a' + rn + b') \mod n \equiv (a' + b') \mod n.$$

2. Similarly

$$(a \mod n)(b \mod n) \equiv a'b' \mod n$$

and

$$(ab) \mod n \equiv (qn+a')(rn+b') \equiv qrn^2+qnb'+a'rn+a'b' \mod n \equiv (a'b') \mod n.$$

Exercise 5. Compute the addition table and the multiplication tables for integers modulo 4.

Solution. We represent integers modulo 4 by the set of integers $\{0, 1, 2, 3\}$. Then

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Similarly

*	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Note that these tables are great to observe the closure property! Elements computed in these tables are the same as those given as input.

Exercise 6. Show that $\frac{p(p+1)}{2} \equiv 0 \pmod{p}$ for p an odd prime.

Solution. Suppose that p is an odd prime. Then $p + 1$ is even, thus divisible by 2, say $p + 1 = 2k$ for some k. Now

$$\frac{p(p+1)}{2} = pk \equiv 0 \pmod{p}.$$

Note that the critical part is that p is odd. If $p = 2$, this does not work, indeed $2 \cdot 3/2 = 3$ which is not $0 \mod 2$.

Exercise 7. Consider the following sets S, with respective operator Δ.

- Let S be the set of rational numbers, and Δ be the multiplication. Is S closed under Δ? Justify your answer.

- Let S be the set of natural numbers, and Δ be the subtraction. Is S closed under Δ? Justify your answer.

- Let S be the set of irrational numbers, and Δ be the addition. Is S closed under Δ? Justify your answer.

Solution.

- Take two rational numbers $\frac{m}{n}$ and $\frac{m'}{n'}$. Then
$$\frac{m}{n}\frac{m'}{n'} = \frac{mm'}{nn'}$$
which is a rational number. Thus the answer is yes, S is closed under multiplication.

- The subtraction of two natural numbers does not always give a number natural, for example,
$$5 - 10 = -5.$$
Thus S is not closed under subtraction.

- The addition of two irrational numbers does not always give an irrational number, for example,
$$(2 + \sqrt{2}) + (2 - \sqrt{2})) = 4$$
and 4 is not an irrational number. Thus S is not closed under addition. Note that we are using here the claim that $2+\sqrt{2}$ is irrational. Indeed, suppose that $2 + \sqrt{2}$ were rational, that is $2 + \sqrt{2} = \frac{m}{n}$ for m, n some integers. Then
$$\sqrt{2} = \frac{m}{n} - 2 = \frac{m - 2n}{n}$$
which is a contradiction to the fact that $\sqrt{2}$ is irrational.

Exercises for Chapter 2

Exercise 8. Decide whether the following statements are propositions. Justify your answer.

1. $2 + 2 = 5$.
2. $2 + 2 = 4$.
3. $x = 3$.
4. Every week has a Sunday.
5. Have you read "Catch 22"?

Solution. 1. $2+2=5$: this is a proposition, because it is a statement that always takes the truth value "false".

2. $2+2=4$: this is a proposition, because it is a statement that always takes the truth value "true".

3. $x=3$: the statement depends on the value of x. Maybe it is true (if x was assigned the value 3), or maybe it is false (if x was assigned a different value). Thus this is not a proposition.

4. Every week has a Sunday: this is a proposition, because it is a statement that always takes the truth value "true".

5. Have you read "Catch 22"?: this is a question, thus it is not a proposition.

Exercise 9. Show that

$$\neg(p \vee q) \equiv \neg p \wedge \neg q.$$

This is the second law of De Morgan.

Solution. We show the equivalence using truth tables:

p	q	$p \vee q$	$\neg(p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Since both truth tables are the same, the two logical expressions are equivalent.

Exercise 10. Show that the second absorption law $p \wedge (p \vee q) \equiv p$ holds.

Solution. We show the equivalence using a truth table:

p	q	$p \vee q$	$p \wedge (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	F

Since the column of p is the same as that of $p \wedge (p \vee q)$, both logical expressions are equivalent.

Exercise 11. These two laws are called distributivity laws. Show that they hold:

1. Show that $(p \wedge q) \vee r \equiv (p \vee r) \wedge (q \vee r)$.
2. Show that $(p \vee q) \wedge r \equiv (p \wedge r) \vee (q \wedge r)$.

Solution. We use truth tables:

p	q	r	$p \wedge q$	$(p \wedge q) \vee r$	$p \vee r$	$q \vee r$	$(p \vee r) \wedge (q \vee r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	F	T	F	F
F	T	T	F	T	T	T	T
F	T	F	F	F	F	T	F
F	F	T	F	T	T	T	T
F	F	F	F	F	F	F	F

p	q	r	$p \vee q$	$(p \vee q) \wedge r$	$p \wedge r$	$q \wedge r$	$(p \wedge r) \vee (q \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	F	T
T	F	F	T	F	F	F	F
F	T	T	T	T	F	T	T
F	T	F	T	F	F	F	F
F	F	T	F	F	F	F	F
F	F	F	F	F	F	F	F

Exercise 12. Verify $\neg(p \vee \neg q) \vee (\neg p \wedge \neg q) \equiv \neg p$ by

- constructing a truth table,
- developing a series of logical equivalences.

Solution. We start with a truth table:

p	q	$\neg p$	$\neg q$	$p \vee \neg q$	$\neg(p \vee \neg q)$	$\neg p \wedge \neg q$	$\neg(p \vee \neg q) \vee (\neg p \wedge \neg q)$
T	T	F	F	T	F	F	F
T	F	F	T	T	F	F	F
F	T	T	F	F	T	F	T
F	F	T	T	T	F	T	T

Next we want to prove this result without using the truth table, but by developing logical equivalences:

$$\begin{aligned}
\neg(p \vee \neg q) \vee (\neg p \wedge \neg q) &\equiv (\neg p \wedge q) \vee (\neg p \wedge \neg q) \text{ De Morgan} \\
&\equiv \neg p \wedge (q \vee \neg q) \text{ Distributivity} \\
&\equiv \neg p \wedge T \text{ since}(q \vee \neg q) \equiv T \\
&\equiv \neg p.
\end{aligned}$$

Exercise 13. Using a truth table, show that:

$$\neg q \to \neg p \equiv p \to q.$$

Solution. We compute the truth table:

p	q	$\neg p$	$\neg q$	$\neg q \to \neg p$	$p \to q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Exercise 14. Show that $p \vee q \to r \equiv (p \to r) \wedge (q \to r)$.

Solution. We use logical equivalences:

$$\begin{aligned}
p \vee q \to r &\equiv (p \vee q) \to r \text{ precedence} \\
&\equiv \neg(p \vee q) \vee r \text{ conversion theorem} \\
&\equiv (\neg p \wedge \neg q) \vee r \text{ De Morgan} \\
&\equiv (\neg p \vee r) \wedge (\neg q \vee r) \text{ Distributivity} \\
&\equiv (p \to r) \wedge (q \to r) \text{ conversion theorem}
\end{aligned}$$

Exercise 15. Are $(p \to q) \vee (q \to r)$ and $p \to r$ equivalent statements?

Solution. They are not equivalent. Here is a proof using truth tables:

p	q	r	$p \to q$	$q \to r$	$(p \to q) \vee (q \to r)$	$p \to r$
T	T	T	T	T	T	T
T	T	F	T	F	T	F
T	F	T	F	T	T	T
T	F	F	F	T	T	F
F	T	T	T	T	T	T
F	T	F	T	F	T	T
F	F	T	T	T	T	T
F	F	F	T	T	T	T

We see that the first rows for example are giving different truth values. This can be done using equivalences as well:

$$\begin{aligned}(p \to q) \vee (q \to r) &\equiv (\neg p \vee q) \vee (\neg q \vee r) \text{ conversion theorem}\\ &\equiv \neg p \vee r \vee T \text{ since } \neg q \vee q \equiv T\\ &\equiv T.\end{aligned}$$

Since $p \to q$ is not equivalent to T, both statements cannot be equivalent.

Exercise 16. Show that this argument is valid:

$$\boxed{\neg p \to F; \therefore p.}$$

Solution. The premise is $\neg p \to F$, which is true when $\neg p$ is false, which is exactly when p is true.

Exercise 17. Show that this argument is valid, where C denotes a contradiction.

$$\boxed{\neg p \to C; \therefore p.}$$

Solution. The premise is $\neg p \to C$, which is true when $\neg p$ is false, which is exactly when p is true.

Exercise 18. Determine whether the following argument is valid:

$$\begin{array}{l}\neg p \to r \wedge \neg s\\ t \to s\\ u \to \neg p\\ \neg w\\ u \vee w\\ \therefore t \to w.\end{array}$$

Solution. We start by noticing that we have

$$u \vee w; \neg w; \therefore u.$$

Indeed, if $u \vee w$ and $\neg w$ are both true, then w is false, and u must be true. Next

$$u \to \neg p; u; \therefore \neg p.$$

Indeed, if $u \to \neg p$ is true, either u is true and $\neg p$ is true, or u is false. But u is true, thus $\neg p$ is true (Modus Ponens). Then

$$\neg p \to r \wedge \neg s; \neg p; \therefore r \wedge \neg s,$$

this is again Modus Ponens. Then

$$r \wedge \neg s; \therefore \neg s.$$

Indeed, for $r \wedge \neg s$ to be true, it must be that $\neg s$ is true. Finally

$$t \to s; \neg s; \therefore \neg t$$

since for $t \to s$ to be true, we need either t to false, or t and s to be true, but since s is false, t must be false (Modus Tollens), and

$$\neg t \therefore \neg t \vee w$$

or equivalently

$$\neg t \vee w \equiv t \to w$$

using the Conversion theorem, which shows that the argument is valid.

Exercise 19. Determine whether the following argument is valid:

$$\begin{array}{l} p \\ p \vee q \\ q \to (r \to s) \\ t \to r \\ \therefore \neg s \to \neg t. \end{array}$$

Solution. For this question, there is no obvious way to combine the known statements with inference rules. The only 2 related statements are p and $p \vee q$, and assuming that both are true, all can be deduced is that q is either true or false. Now if q is false, $q \to (r \to s)$ is always true, while if q is true, $q \to (r \to s)$ is true only if $(r \to s)$ is true, which excludes the possibility $r = T$ and $s = F$.

q	r	s	t
F	T	T	
F	T	F	
F	F	T	
F	F	F	
F	T	T	
T	F	T	
T	F	F	

Now we look at the last premise $t \to r$. For it to be true, we need t false, or t true and r true.

q	r	s	t
F	T	T	T/F
F	T	F	T/F
F	F	T	F
F	F	F	F
T	T	T	T/F
T	F	T	F
T	F	F	F

Now if s is true, then $\neg s$ is always false, and the conclusion is always true. We thus focus on s is false, and $\neg t$ is false, that is t is true. The second row gives a counter-example:

$$q = F,\ r = T,\ s = F,\ t = T.$$

Exercises for Chapter 3

Exercise 20. Consider the predicates $M(x, y) =$ "x has sent an email to y", and $T(x, y) =$ "x has called y". The predicate variables x, y take values in the domain $D = \{$students in the class$\}$. Express these statements using symbolic logic.

1. There are at least two students in the class such that one student has sent the other an email, and the second student has called the first student.

2. There are some students in the class who have emailed everyone.

Solution. 1. We need two predicate variables since at least 2 students are involved, say x and y. There are at least two students in the class becomes

$$\exists x \in D,\ \exists y \in D.$$

Then x sent an email to y, that is $M(x,y)$ and y has called x, that is $T(y,x)$, thus

$$M(x,y) \wedge T(y,x).$$

Furthermore, we need to take into account the fact that there are at least "two" students, so x and y have to be distinct! Thus the final answer is

$$\exists x \in D,\ \exists y \in D,\ ((x \neq y) \wedge M(x,y) \wedge T(y,x)).$$

2. There are students becomes

$$\exists x \in D,$$

then x has emailed everyone, that is

$$\exists x \in D, (\forall y \in D\ M(x,y)).$$

Note that the order of the quantifiers is important.

Exercise 21. Consider the predicate $C(x,y) =$ "x is enrolled in the class y", where x takes values in the domain $S = \{\text{students}\}$, and y takes values in the domain $D = \{\text{courses}\}$. Express each statement by an English sentence.

1. $\exists\ x \in S,\ C(x, \text{MH1812})$.
2. $\exists\ y \in D,\ C(\text{Carol}, y)$.
3. $\exists\ x \in S,\ (C(x, \text{MH1812}) \wedge C(x, \text{CZ2002}))$.
4. $\exists\ x \in S,\ \exists\ x' \in S,\ \forall y \in D,\ ((x \neq x') \wedge (C(x,y) \leftrightarrow C(x',y)))$.

Solution. 1. There exists a student such that this student is enrolled in the class MH1812, that is some student enrolled in the class MH1812.

2. There exists a course such that Carol is enrolled in this course, that is, Carol is enrolled in some course, or Carol is enrolled in at least one course.

3. There exists a student, such that this student is enrolled in MH1812 and in CZ2002, that is some student is enrolled in both MH1812 and CZ2002.

4. There exist two distinct students x and x', such that for all courses, x is enrolled in the course if and only if x' is enrolled in the course. In other words, there exist two students which are enrolled in exactly the same courses.

Exercise 22. Consider the predicate $P(x, y, z) =$ "$xyz = 1$", for $x, y, z \in \mathbb{R}$, $x, y, z > 0$. What are the truth values of these statements? Justify your answer.

1. $\forall\ x,\ \forall\ y,\ \forall\ z,\ P(x, y, z)$.
2. $\exists\ x,\ \exists\ y,\ \exists\ z,\ P(x, y, z)$.
3. $\forall\ x,\ \forall\ y,\ \exists\ z,\ P(x, y, z)$.
4. $\exists\ x,\ \forall\ y,\ \forall\ z,\ P(x, y, z)$.

Solution.

1. $\forall\ x,\ \forall\ y,\ \forall\ z,\ P(x, y, z)$ is false: take $x = 1$ and $y = 1$, then whenever $z \neq 1$, $xyz = z \neq 1$.
2. $\exists\ x,\ \exists\ y,\ \exists\ z,\ P(x, y, z)$ is true: take $x = y = z = 1$.
3. $\forall\ x,\ \forall\ y,\ \exists\ z,\ P(x, y, z)$ is true: choose any x and any y, then there exists a z, namely $z = \frac{1}{xy}$ such that $xyz = 1$.
4. $\exists\ x,\ \forall\ y,\ \forall\ z,\ P(x, y, z)$ is false: one cannot find a single x such that $xyz = 1$ no matter what are y and z. This is because once yz are chosen, then x is completely determined, so x changes whenever yz does.

Exercise 23.

1. Express
$$\neg(\forall\ x,\ \forall\ y,\ P(x, y))$$
in terms of existential quantification.

2. Express
$$\neg(\exists\ x,\ \exists\ y,\ P(x, y))$$
in terms of universal quantification.

Solution. 1. We see that $\neg(\forall\ x,\ \forall\ y,\ P(x,y))$ is a negation of two universal quantifications. Denote $Q(x)$ = "$\forall\ y,\ P(x,y)$", then $\neg(\forall x, Q(x))$ is $(\exists x, \neg Q(x))$, thus

$$\neg(\forall\ x,\ \forall\ y,\ P(x,y)) \equiv \exists x, \neg(\forall\ y,\ P(x,y))$$

and now we iterate the same rule on the next negation, to get

$$\neg(\forall\ x,\ \forall\ y,\ P(x,y)) \equiv \exists x,\ \exists y,\ \neg P(x,y).$$

2. We repeat the same procedure with the negation of two existential quantifications, by setting this time $Q(x)$ = "$\exists\ y,\ P(x,y)$":

$$\begin{aligned} \neg(\exists\ x,\ \exists\ y,\ P(x,y)) &\equiv \neg(\exists x Q(x)) \\ &\equiv \forall x \neg Q(x) \\ &\equiv \forall x \neg(\exists\ y,\ P(x,y)) \\ &\equiv \forall x \forall y \neg P(x,y). \end{aligned}$$

Exercise 24. Consider the predicate $C(x,y)$ = "x is enrolled in the class y", where x takes values in the domain $S = \{\text{students}\}$, and y takes values in the domain $C = \{\text{courses}\}$. Form the negation of these statements:

1. $\exists\ x,\ (C(x, \text{MH1812}) \wedge C(x, \text{CZ2002}))$.
2. $\exists\ x\ \exists\ y,\ \forall z,\ ((x \neq y) \wedge (C(x,z) \leftrightarrow C(y,z)))$.

Solution. 1. We have

$$\begin{aligned} &\neg(\exists\ x,\ (C(x, \text{MH1812}) \wedge C(x, \text{CZ2002}))) \\ \equiv\ &\forall x \neg(C(x, \text{MH1812}) \wedge C(x, \text{CZ2002})) \\ \equiv\ &\forall x \neg C(x, \text{MH1812}) \vee \neg C(x, \text{CZ2002}) \end{aligned}$$

where the first equivalence is the negation of quantification, and the second equivalence De Morgan's law.

2. We have

$$\begin{aligned} &\neg(\exists\ x\ \exists\ y,\ \forall z,\ ((x \neq y) \wedge (C(x,z) \leftrightarrow C(y,z)))) \\ \equiv\ &\forall x \neg(\exists\ y,\ \forall z,\ ((x \neq y) \wedge (C(x,z) \leftrightarrow C(y,z)))) \\ \equiv\ &\forall x \forall y \neg(\forall z,\ ((x \neq y) \wedge (C(x,z) \leftrightarrow C(y,z)))) \\ \equiv\ &\forall x \forall y \exists z \neg((x \neq y) \wedge (C(x,z) \leftrightarrow C(y,z))) \\ \equiv\ &\forall x \forall y \exists z \neg(x \neq y) \vee \neg(C(x,z) \leftrightarrow C(y,z))) \end{aligned}$$

using three times the negation of quantification, and lastly the Morgan's law. Next $\neg(x \neq y) = (x = y)$ and using that

$$C(x,z) \leftrightarrow C(y,z) \equiv (C(x,z) \rightarrow C(y,z)) \wedge (C(y,z) \rightarrow C(x,z))$$

we get

$$\neg(C(x,z) \leftrightarrow C(y,z)) \equiv \neg(C(x,z) \rightarrow C(y,z)) \vee \neg(C(y,z) \rightarrow C(x,z))$$

so that, using the Conversion theorem to get $\neg(\neg C(x,z) \vee C(y,z))$ and $\neg(\neg C(y,z) \vee C(x,z))$

$$\begin{aligned} & \neg(\exists\, x\ \exists\, y,\ \forall z,\ ((x \neq y) \wedge (C(x,z) \leftrightarrow C(y,z)))) \\ \equiv\ & \forall x \forall y \exists z((x = y) \vee [(C(x,z) \wedge \neg C(y,z)) \vee (C(y,z) \wedge \neg C(x,z))]). \end{aligned}$$

The last term can be further modified using distributivity:

$$\begin{aligned} & (C(x,z) \wedge \neg C(y,z)) \vee (C(y,z) \wedge \neg C(x,z)) \\ \equiv\ & [(C(x,z) \wedge \neg C(y,z)) \vee C(y,z)] \wedge [(C(x,z) \wedge \neg C(y,z)) \vee \neg C(x,z)] \\ \equiv\ & (C(x,z) \vee C(y,z)) \wedge (\neg C(x,z) \vee \neg C(y,z)) \end{aligned}$$

to finally get

$$\begin{aligned} & \neg(\exists\, x\ \exists\, y,\ \forall z,\ ((x \neq y) \wedge (C(x,z) \leftrightarrow C(y,z)))) \\ \equiv\ & \forall x \forall y \exists z((x = y) \vee [(C(x,z) \vee C(y,z)) \wedge (\neg C(x,z) \vee \neg C(y,z))]). \end{aligned}$$

When many steps are involved, it is often a good idea to check the sanity of the answer. If we look at $\neg(C(x,z) \leftrightarrow C(y,z))$, it is false exactly when $C(x,z)$ and $C(y,z)$ are taking the same truth value (either both true or both false). Now we look at $(C(x,z) \vee C(y,z)) \wedge (\neg C(x,z) \vee \neg C(y,z))$: when $C(x,z)$ and $C(y,z)$ are taking the same value, we get false, and true otherwise. This makes sense!

Exercise 25. Show that $\forall x \in D,\ P(x) \rightarrow Q(x)$ is equivalent to its contrapositive.

Solution. For every instantiation of x, $(\forall x \in D,\ P(x) \rightarrow Q(x))$ is a proposition, thus we can use the conversion theorem:

$$\begin{aligned} & (\forall x \in D,\ P(x) \rightarrow Q(x)) \\ \equiv\ & (\forall x \in D, \neg P(x) \vee Q(x)) \\ \equiv\ & (\forall x \in D, Q(x) \vee \neg P(x)) \\ \equiv\ & (\forall x \in D, \neg\neg Q(x) \vee \neg P(x)) \\ \equiv\ & (\forall x \in D, \neg Q(x) \rightarrow \neg P(x)). \end{aligned}$$

Exercise 26. Show that

$$\neg(\forall\ x, P(x) \to Q(x)) \equiv \exists x,\ P(x) \wedge \neg Q(x).$$

Solution.

$$\begin{aligned} & \neg(\forall\ x, P(x) \to Q(x)) \\ \equiv\ & \exists x,\ \neg(P(x) \to Q(x)) \\ \equiv\ & \exists x,\ \neg(\neg P(x) \vee Q(x)) \\ \equiv\ & \exists x,\ (P(x) \wedge \neg Q(x)) \end{aligned}$$

where the first equivalence is the negation of quantifications, the second equivalence is the conversion theorem, and the third one is De Morgan's law.

Exercise 27. Let y, z be positive integers. What is the truth value of "$\exists y,\ \exists z,\ (y = 2z \wedge (y \text{ is prime}))$".

Solution. The truth value is true, take $y = 2$ and $z = 1$.

Exercise 28. Write in symbolic logic "Every SCE student studies discrete mathematics. Jackson is an SCE student. Therefore Jackson studies discrete mathematics".

Solution. Consider the domain $D = \{$ SCE students $\}$. Set $P(x)$="x studies discrete mathematics. Then every SCE student studies discrete mathematics becomes

$$\forall x \in D,\ P(x).$$

Now Jackson is a SCE student means Jackson belongs to D. This gives

$$\forall x \in D,\ P(x); \text{Jackson} \in D; \therefore P(\text{Jackson}).$$

Exercise 29. Here is an optional exercise about universal generalization. Consider the following two premises: (1) for any number x, if $x > 1$ then $x - 1 > 0$, (2) every number in D is greater than 1. Show that therefore, for every number x in D, $x - 1 > 0$.

Solution. Set $P(x)$ = "$x > 1$" and $Q(x)$ = "$x - 1 > 0$". Let us formalize what we want to prove:

$$[\forall x\ (P(x) \to Q(x)) \wedge \forall x \in D\ P(x)] \to \forall x \in D,\ Q(x).$$

1. $\forall x\ (P(x) \rightarrow Q(x))$, by hypothesis
2. $\forall x \in D\ P(x)$, also by hypothesis
3. $P(y) \rightarrow Q(y)$, by universal instantiation on the first hypothesis
4. $P(y)$, by universal instantiation on D in the second hypothesis
5. $Q(y)$, using modus ponens
6. $\forall x \in D,\ Q(x)$, using universal generalization.

Exercise 30. Let q be a positive real number. Prove or disprove the following statement: if q is irrational, then $\sqrt{q}$ is irrational.

Solution. We prove the contrapositive of this statement, namely: if $\sqrt{q}$ is rational, then q is rational. But if $\sqrt{q}$ is rational, then $\sqrt{q} = \frac{a}{b}$, a, b integers, $b \neq 0$, and thus $q = \frac{a^2}{b^2}$ which shows that q is rational.

Exercise 31. Prove using mathematical induction that the sum of the first n odd positive integers is n^2.

Solution. We want to prove that $\forall n,\ P(n)$ where

$$P(n) = \text{``}\sum_{j=1}^{n}(2j-1) = n^2\text{''},\ n \in \mathbb{Z},\ \geq 1.$$

- Basis Step: we need to show that $P(1)$ is true.

$$P(1) = (2-1) = 1 = 1^2$$

  which is true.

- Inductive Step: Assume $P(k)$ is true, that is we assume that

$$\sum_{j=1}^{k}(2j-1) = k^2$$

is true. We now need to prove that $P(k+1)$ is true.

$$\begin{aligned}&\sum_{j=1}^{k+1}(2j-1)\\ =\ &\sum_{j=1}^{k}(2j-1)+2(k+1)-1\\ =\ &k^2+2(k+1)-1 \text{ using } P(k)\\ =\ &k^2+2k+1=(k+1)^2.\end{aligned}$$

This shows that $P(k+1)$ is true, therefore $P(n)$ is true for all n.

Exercise 32. Prove using mathematical induction that n^3-n is divisible by 3 whenever n is a positive integer.

Solution. We first set $P(n)=$ "$3 \mid n^3-n$", that is 3 divides n^3-n.

- Basis Step: $P(1)=$ "$3|0$" which is true.
- Inductive Step: Assume $P(k)$ is true, that is we assume that

  $$3 \mid (k^3-k).$$

  is true. We now need to prove that $P(k+1)$ is true. When $n=k+1$, we get

  $$(k+1)^3-(k+1)=(k^3+3k^2+3k+1)-(k+1)=(k^3-k)+3k^2+3k$$

  which is divisible by 3, since (k^3-k) is divisible by 3, and so is $3(k^2+k)$. Therefore $P(k+1)$ is true, and we conclude that $P(n)$ is true for all n.

Exercises for Chapter 4

Exercise 33. 1. Show that

$$\binom{n}{k}+\binom{n}{k-1}=\binom{n+1}{k}$$

for $1 \leq k \leq l$, where by definition

$$\binom{n}{k}=\frac{n!}{k!(n-k)!},\ n!=n\cdot(n-1)\cdot(n-2)\cdots 2\cdot 1.$$

2. Prove by mathematical induction that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

   You will need 1. for this!

3. Deduce that the cardinality of the power set $P(S)$ of a finite set S with n elements is 2^n.

Solution. To prove

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k},$$

we first expand the left hand side:

$$\frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} = \frac{n!(n-k+1)}{k!(n-k+1)!} + \frac{n!k}{k!(n-k+1)!}$$

which is equal to

$$\frac{n!(n+1)}{k!(n-k+1)!} = \frac{(n+1)!}{k!(n-k+1)!} = \binom{n+1}{k}.$$

To prove the binomial theorem by mathematical induction, we set

$$P(n) = \text{``}(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k\text{''},$$

and we want to prove that $\forall n,\ P(n)$. The basis step is to prove that $P(1)$ holds, which is given by

$$(x+y) = \sum_{k=0}^{1} \binom{1}{k} x^{1-k} y^k = x+y.$$

Next for the inductive step, we suppose that $P(l)$ is true, namely

$$(x+y)^l = \sum_{k=0}^{l} \binom{l}{k} x^{l-k} y^k \tag{12.1}$$

and we want to prove $P(l+1)$.

$$\begin{aligned}
(x+y)^{l+1} &= (x+y)(x+y)^l \\
&= (x+y)\sum_{k=0}^{l}\binom{l}{k}x^{l-k}y^k \text{ (using (12.1))} \\
&= x\sum_{k=0}^{l}\binom{l}{k}x^{l-k}y^k + y\sum_{k=0}^{l}\binom{l}{k}x^{l-k}y^k \\
&= \sum_{k=0}^{l}\binom{l}{k}x^{l-k+1}y^k + \sum_{k=0}^{l}\binom{l}{k}x^{l-k}y^{k+1}.
\end{aligned}$$

At this point, it is probably a good idea to remember what we want to prove, namely

$$(x+y)^{l+1} = \sum_{k=0}^{l+1}\binom{l+1}{k}x^{(l+1)-k}y^k.$$

From our aim, we notice that the first sum has already right exponents, namely $x^{l-k+1}y^k$ is a term we want. So we first work on the other sum to get a similar right term present, by doing a change of variable $j = k+1$, to get

$$\sum_{k=0}^{l}\binom{l}{k}x^{l-k}y^{k+1} = \sum_{j=1}^{l+1}\binom{l}{j-1}x^{l-j+1}y^j.$$

We next combine this derivation:

$$\begin{aligned}
&(x+y)^{l+1} \\
=\ & \sum_{k=0}^{l}\binom{l}{k}x^{l-k+1}y^k + \sum_{j=1}^{l+1}\binom{l}{j-1}x^{l-j+1}y^j \\
=\ & \sum_{k=1}^{l}\binom{l}{k}x^{l-k+1}y^k + \binom{l}{0}x^{l+1} + \sum_{j=1}^{l}\binom{l}{j-1}x^{l-j+1}y^j + \binom{l}{l}y^{l+1} \\
=\ & \sum_{k=1}^{l}\left[\binom{l}{k} + \binom{l}{k-1}\right]x^{l-k+1}y^k + y^{l+1} + x^{l+1}
\end{aligned}$$

and now is the point where we recognize the formula that we derived in 1., thus

$$\begin{aligned}&(x+y)^{l+1}\\ =\;& \sum_{k=1}^{l}\binom{l+1}{k}x^{l-k+1}y^k+y^{l+1}+x^{l+1}\\ =\;& \sum_{k=0}^{l+1}\binom{l+1}{k}x^{l-k+1}y^k\end{aligned}$$

Finally, evaluate the binomial theorem in $x=y=1$. The only thing left to be seen is the interpretation of $\binom{n}{k}$ as "n choose k", which will be discussed into more details in the next chapter, namely $\binom{n}{k}$ counts the number of ways of picking k elements out of n. Therefore to count the number of elements in $P(S)$ we just count how many subsets we have with 1 element, with 2 elements, ..., and we sum these numbers up!

Exercise 34. Let $P(C)$ denote the power set of C. Given $A=\{1,2\}$ and $B=\{2,3\}$, determine:

$$P(A\cap B),\ P(A),\ P(A\cup B),\ P(A\times B).$$

Solution.

- $A\cap B=\{2\}$, therefore $P(A\cap B)=\{\varnothing,\{2\}\}$.
- $P(A)=\{\varnothing,\{1\},\{2\},\{1,2\}\}$.
- $A\cup B=\{1,2,3\}$, therefore $P(A\cup B)=\{\varnothing,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$.
- $A\times B=\{(1,2),(1,3),(2,2),(2,3)\}$, therefore $P(A\times B)$ contains
  - $\varnothing$, $\{(1,2)\}$,$\{(1,3)\}$,$\{(2,2)\}$,$\{(2,3)\}$,
  - $\{(1,2),(1,3)\}$, $\{(1,2),(2,2)\}$, $\{(1,2),(2,3)\}$,$\{(1,3),(2,2)\}$, $\{(1,3),(2,3)\}$, $\{(2,2),(2,3)\}$,
  - $\{(1,2),(1,3),(2,2)\}$,$\{(1,2),(1,3),(2,3)\}$,$\{(1,2),(2,3),(2,2)\}$,$\{(1,3),(2,2),(2,3)\}$,
  - $\{(1,2),(1,3),(2,2),(2,3)\}$.

Exercise 35. Prove by contradiction that for two sets A and B

$$(A-B)\cap(B-A)=\varnothing.$$

Solution. Suppose by contradiction that $(A - B) \cap (B - A)$ is not empty. Then there exists an element x which belongs to both $(A-B)$ and $(B-A)$. This means that x belongs to A (since $x \in A - B$), and x does not belong to A (since $x \in B - A$), which is a contradiction! Therefore the assumption was false, and $(A - B) \cap (B - A)$ is empty.

Note that from a propositional logic point of view, what we did is set p="$(A - B) \cap (B - A) = \varnothing$", q="$x \in A$", and we prove that

$$\neg p \to (q \wedge \neg q)$$

which turns out to be equivalent to p.

Exercise 36. Let $P(C)$ denote the power set of C. Prove that for two sets A and B

$$P(A) = P(B) \iff A = B.$$

Solution. We need to show that $P(A) = P(B) \to A = B$ and $A = B \to P(A) = P(B)$.

- suppose $P(A) = P(B)$: then all sets containing one element are the same for $P(A)$ and $P(B)$, and $A = B$.
- suppose $A = B$: subsets of A and subsets of B are the same, and $P(A) = P(B)$.

Exercise 37. Let $P(C)$ denote the power set of C. Prove that for two sets A and B

$$P(A) \subseteq P(B) \iff A \subseteq B.$$

Solution. We need to show that $P(A) \subseteq P(B) \to A \subseteq B$ and $A \subseteq B \to P(A) \subseteq P(B)$.

- suppose $P(A) \subseteq P(B)$: then $A \subseteq P(B)$, from which $A \subseteq B$.
- suppose $A \subseteq B$: then for any $X \in P(A)$, $X \subseteq A$, $X \subseteq B$, therefore $X \in P(B)$.

Exercise 38. Show that the empty set is a subset of all non-null sets.

Solution. Recall the definition of subset: $Y \subseteq X$ means by definition that $\forall x$, $(x \in Y \to x \in X)$. Now take Y to be the empty set $\varnothing$. Since $x \in Y$ is necessarily false (one cannot take any x in the empty set), then the conditional statement is vacuously true.

Exercise 39. Show that for two sets A and B

$$A \neq B \equiv \exists x[(x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)].$$

Solution.

$$\begin{aligned} A \neq B &= \neg \forall x (x \in A \leftrightarrow x \in B) \\ &\equiv \exists x \neg (x \in A \leftrightarrow x \in B) \text{ (negation of universal quantifier)} \\ &\equiv \exists x \neg [(x \in A \to x \in B) \wedge (x \in B \to x \in A)] \text{(definition)} \\ &\equiv \exists x [\neg (x \in A \to x \in B) \vee \neg (x \in B \to x \in A)] \text{(DeMorgan)} \\ &\equiv \exists x [\neg (x \notin A \vee x \in B) \vee \neg (x \notin B \vee x \in A)] \text{ (Conversion)} \\ &\equiv \exists x [(x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)] \text{ (DeMorgan)} \end{aligned}$$

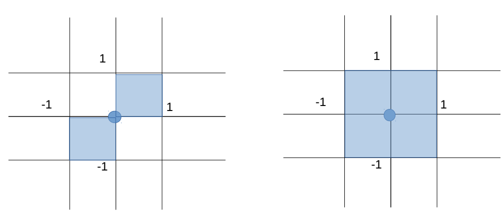
Exercise 40. Prove that for the sets A, B, C, D

$$(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D).$$

Does equality hold?

Solution. Suppose $x \in (A \times B) \cup (C \times D)$, then $x = (x_1, x_2)$ with $x_1 \in A$, $x_2 \in B$ (or $x_1 \in C$, $x_2 \in D$). But then $x_1 \in A$ (or C), and $x_2 \in B$ (or D), therefore $x \in (A \cup C) \times (B \cup D)$. The equality does not hold: take $A = [-1, 0]$, $B = [-1, 0]$, $C = [0, 1]$, $D = [0, 1]$ (all the set are intervals, that is $[a, b]$ means the interval from a to b). Then

$$([-1, 0] \times [-1, 0]) \cup ([0, 1] \times [0, 1]) \neq [-1, 1] \times [-1, 1].$$



Exercise 41. Does the equality

$$(A_1 \cup A_2) \times (B_1 \cup B_2) = (A_1 \times B_1) \cup (A_2 \times B_2)$$

hold?

Solution. No it does not. For example, take $A_1 = \{0\}$, $A_2 = \{1\}$, $B_1 = \{0\}$, $B_2 = \{1\}$, then

$$\{0,1\} \times \{0,1\} \neq \{(0,0),(1,1)\}.$$

Exercise 42. For all sets A, B, C, prove that

$$\overline{(A - B) - (B - C)} = \bar{A} \cup B.$$

using set identities.

Solution. We have

$$\begin{aligned}
\overline{(A - B) - (B - C)} &= \overline{(A \cap \bar{B}) - (B \cap \bar{C})} \\
&= \overline{(A \cap \bar{B}) \cap \overline{(B \cap \bar{C})}} \\
&= \overline{(A \cap \bar{B})} \cup (B \cap \bar{C}) \\
&= (\bar{A} \cup B) \cup (B \cap \bar{C}) \\
&= \bar{A} \cup B \cup (B \cap \bar{C}) \\
&= \bar{A} \cup B
\end{aligned}$$

Both the 3rd and 4rth equality follows from De Morgan's Laws for sets, and $\bar{\bar{S}} = S$ for any set S. The 5th equality is associativity, while the last equality is true because $(B \cap \bar{C})$ is a subset of B.

Exercise 43. This exercise is more difficult. For all sets A and B, prove $(A \cup B) \cap \overline{A \cap B} = (A - B) \cup (B - A)$ by showing that each side of the equation is a subset of the other.

Solution. We have to prove that (1) $(A \cup B) \cap \overline{A \cap B} \subseteq (A - B) \cup (B - A)$ and (2) $(A \cup B) \cap \overline{A \cap B} \supseteq (A - B) \cup (B - A)$.

Part (1). Suppose that $x \in (A \cup B) \cap \overline{A \cap B}$, then

$$(x \in (A \cup B)) \wedge (x \in \overline{A \cap B}).$$

Now $(x \in (A \cup B))$ means that $(x \in A) \vee (x \in B)$, that is

$$\begin{aligned} & [(x \in A) \vee (x \in B)] \wedge (x \in \overline{A \cap B}) \\ \equiv\ & [(x \in A) \wedge (x \in \overline{A \cap B})] \vee [(x \in B) \wedge (x \in \overline{A \cap B})] \end{aligned}$$

using distributivity.

Next $(x \in \overline{A \cap B})$ means that $x \notin A \cap B$ (x always lives in the universe U, so it is not repeated). Now the negation of $x \in A \cap B$ is $(x \notin A) \vee (x \notin B)$. We thus get that $(x \in A) \wedge (x \in \overline{A \cap B})$ becomes

$$(x \in A) \wedge [(x \notin A) \vee (x \notin B)] \equiv F \vee [(x \in A) \wedge (x \notin B)]$$

using distributivity. Repeating the same procedure by flipping the role of B and A in $(x \in B) \wedge (x \in \overline{A \cap B})$, we finally obtain that

$$[(x \in A) \wedge (x \notin B)] \vee [(x \in B) \wedge (x \notin A)].$$

We have thus shown that $(A \cup B) \cap \overline{A \cap B} \subseteq (A - B) \cup (B - A)$.

Part (2). For the second part, we need to show $(A \cup B) \cap \overline{A \cap B} \supseteq (A - B) \cup (B - A)$.

Suppose thus that $x \in (A - B) \cup (B - A)$, that is $x \in (A - B)$ or $x \in (B - A)$. If $x \in (A - B)$ then $x \in A$ and $x \notin B$ by definition. Therefore $x \in A \cup B$ and $x \notin A \cap B$.

Similarly, if $x \in (B - A)$ then $x \in B$ and $x \notin A$ by definition. Therefore $x \in A \cup B$ and $x \notin A \cap B$.

Exercise 44. The symmetric difference of A and B, denoted by $A \oplus B$, is the set containing those elements in either A or B, but not in both A and B.

1. Prove that $(A \oplus B) \oplus B = A$ by showing that each side of the equation is a subset of the other.

2. Prove that $(A \oplus B) \oplus B = A$ using a membership table.

Solution. It is a good idea to draw a Venn diagram to visualize $(A \oplus B)$, which consists of $A \cup B$ without the intersection $A \cap B$.

1. We have to show that (1) $(A \oplus B) \oplus B \subseteq A$, and (2) $(A \oplus B) \oplus B \supseteq A$.

   - $(A \oplus B) \oplus B \subseteq A$: take $x \in (A \oplus B) \oplus B$. If $x \in B$, then $x \notin A \oplus B$ by definition. But then, it must be that $B \in A \cap B$ that is, $x \in A$ as desired. Next if $x \notin B$, then $x \in A \oplus B$ by definition. But then x is in the union $A \cup B$ though not in the intersection $A \cap B$, and since it is not in B, it must be in A.

- $(A \oplus B) \oplus B \supseteq A$: we now start with $x \in A$. If $x \in B$ (that is, $x \in A \cap B$), then $x \notin A \oplus B$, then $x \in (A \oplus B) \oplus B$. Next if $x \notin B$, then $x \in A \oplus B$, and thus $x \in (A \oplus B) \oplus B$.

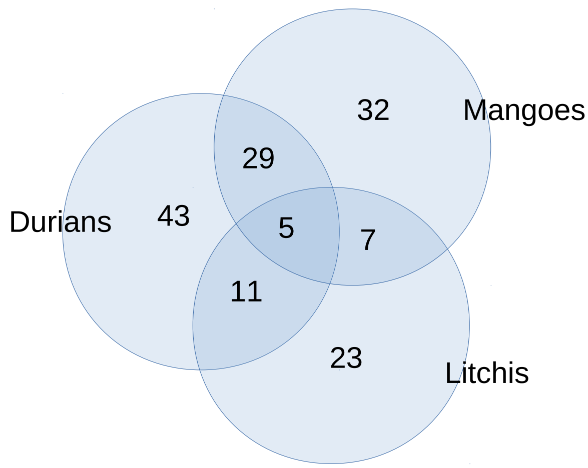
2. We construct a membership table as shown next:

A	B	$A \oplus B$	$(A \oplus B) \oplus B$
0	0	0	0
0	1	1	0
1	0	1	1
1	1	0	1

For example, for the second row, $x \in B$ but not in A. Then $x \in A \oplus B$. But then it cannot be in $(A \oplus B) \oplus B$ since x is in both B and $A \oplus B$. We conclude since both the first and last column are the same.

Exercise 45. In a fruit feast among 200 students, 88 chose to eat durians, 73 ate mangoes, and 46 ate litchis. 34 of them had eaten both durians and mangoes, 16 had eaten durians and litchis, and 12 had eaten mangoes and litchis, while 5 had eaten all 3 fruits. Determine, how many of the 200 students ate none of the 3 fruits, and how many ate only mangoes?

Solution. Let us draw a Venn diagram with 3 sets (one for each of the fruits) and start by identifying the numbers of students who ate all the 3 fruits, namely 5 of them. Then we identify the number of students who ate two types of fruits (for example, 34 ate durians and mangoes, so 34-5=29), and finally only one type of fruit.



We thus get a total of 150, meaning that 50 students ate nothing. 32 students ate only mangoes.

Exercises for Chapter 5

Exercise 46. A set menu proposes 2 choices of starters, 3 choices of main dishes, and 2 choices of desserts. How many possible set menus are available?

Solution. You have 2 choices of starters, then for any choice, you get 3 choices of main dishes, or for each of them you get 2 choices of desserts. Therefore the total is

$$2 \cdot 3 \cdot 2 = 12.$$

Exercise 47. Consider the set $A = \{1, 2, 3\}$, $P(A)$ =power set of A.

- Compute the cardinality of $P(A)$ using the binomial theorem approach.
- Compute the cardinality of $P(A)$ using the counting approach.

Solution.
- To compute the cardinality of $P(A)$, we need to count the empty set (1), the number of subset of size 1 ($\binom{3}{1}$), the number of sets

of size 2 ($\binom{3}{2}$) and the whole set (1), therefore:

$$|P(A)| = 1 + \binom{3}{1} + \binom{3}{2} + 1 = 2^3$$

where the last equality follows from the binomial theorem.

- In a counting approach, we write binary strings to identify whether a given element belongs to a subset, for example 000 corresponds to the empty set, and 000 is read as 1 is not in this subset, 2 is not, and 3 is not either. Now for every subset, each element either belongs, or does not belong, therefore we get the 8 possible binary strings, and the cardinality is 2^3.

Exercise 48.

- Two fair coins are tossed. What is the probaiblity of getting 2 heads?
- Three fair coins are tossed. What is the probability of getting exactly 2 heads?

Solution.

- Two fair coins are tossed, therefore the sample space is

$$\{HH, HT, TH, TT\}.$$

The probability of getting HH is therefore 1/4.

- Three fair coins are tossed, therefore the sample space is

$$\{HHH, HTH, THH, TTH, HHT, HTT, THT, TTT\}.$$

The event getting exactly 2 heads is $\{HHT, HTH, THH\}$. Thefore the probability is 3/8.

Exercise 49. Ten fair coins are tossed together. What is the probability that there were at least seven heads?

Solution. To have at least seven heads means that the number of heads is either 7, 8, 9 or 10. The number of patterns for 7 heads is $C(10, 7)$, the number of patterns for 8 heads is $C(10, 8)$, the number of patterns for 9 heads is $C(10, 9)$, and finally $C(10, 10)$ for 10 heads. The total number of outcomes is 2^{10}, thus we get

$$\frac{\sum_{i=7}^{10} C(10, i)}{2^{10}} = \frac{120 + 45 + 10 + 1}{2^{10}} = \frac{176}{2^{10}}$$

since

$$C(10,7) = \frac{10!}{7!3!} = 10 \cdot 3 \cdot 4,\ C(10,8) = \frac{10!}{8!2!} = 5 \cdot 9,\ C(10,9) = \frac{10!}{9!} = 10.$$

Exercise 50. Snow white is going to a party with the seven dwarves. Each of the eight of them owns a red dress and a a blue dress. If each of them is likely to choose either colored dress randomly and independently of the other's choices, what is the chance that all of them go to the pary wearing the same colored dress?

Solution. Either they all dress in blue, or they all dress in blue, therefore

$$\frac{2}{2^8} = \frac{1}{2^7}.$$

Exercises for Chapter 6

Exercise 51. Consider the linear recurrence $a_n = 2a_{n-1} - a_{n-2}$ with initial conditions $a_1 = 3$, $a_0 = 0$.

- Solve it using the backtracking method.
- Solve it using the characteristic equation.

Solution.

- We have $a_n = 2a_{n-1} - a_{n-2}$, thus $a_{n-1} = 2a_{n-2} - a_{n-3}$, $a_{n-2} = 2a_{n-3} - a_{n-4}$, $a_{n-3} = 2a_{n-4} - a_{n-5}$, etc therefore

$$\begin{aligned} a_n &= 2a_{n-1} - a_{n-2} \\ &= 2(2a_{n-2} - a_{n-3}) - a_{n-2} = 3a_{n-2} - 2a_{n-3} \\ &= 3(2a_{n-3} - a_{n-4}) - 2a_{n-3} = 4a_{n-3} - 3a_{n-4} \\ &= 4(2a_{n-4} - a_{n-5}) - 3a_{n-4} = 5a_{n-4} - 4a_{n-5} \\ &= \dots \end{aligned}$$

We see that a general term is $(i+1)a_{n-i} - ia_{n-(i+1)}$. Therefore the last term is when $n - i - 1 = 0$ that is $i = n - 1$, for which we have $na_1 - (n-1)a_0$, therefore with initial condition $a_0 = 0$ and $a_1 = 3$, we get

$$a_n = 3n.$$

Optional. Now if one wants to be sure that this is indeed the right answer, this can be checked using a proof by mathematical induction! However here, the mathematical induction is slightly different from our usual one! We have

$$P(n) = \text{“}a_n = 3n\text{”},$$

so the basis step which is $P(0) =$ “$a_0 = 0$” holds. However we will also need a second basis step, which is $P(1) =$ “$a_1 = 3$”, which still holds. Now suppose $P(k) =$ “$a_k = 3k''$ and $P(k-1) =$ “$a_{k-1} = 3(k-1)''$ are both true. Then

$$\begin{aligned} a_{k+1} &= 2a_k - a_{k-1} \\ &= 6k - 3(k-1) \\ &= 6k - 3k + 3 = 3k + 3 = 3(k+1) \end{aligned}$$

as needed, where we used both our induction hypotheses!

- Suppose now we want to solve the same recurrence using a characteristic equation. We have $x^n = 2x^{n-1} - x^{n-2}$ that is

  $$x^n - 2x^{n-1} + x^{n-2} = 0 \iff x^{n-2}(x^2 - 2x + 1) = 0.$$

  We have $x^2 - 2x + 1 = (x-1)^2$, therefore

  $$a_n = u + vn.$$

  Then

  $$a_0 = u = 0,\ a_1 = u + v = 3$$

  thus $v = 3$, yielding

  $$a_n = 3n.$$

Exercise 52. What is the solution of the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}$$

with $a_0 = 2$ and $a_1 = 7$?

Solution. The characteristic equation is $x^2 - x - 2 = 0$. Its roots are $x = -1$ and $x = 2$ since $(x+1)(x-2) = 0$. Therefore $a_n = u2^n + v(-1)^n$ is a solution. We are left with identifying u, v using the initial conditions.

$$a_0 = 2 = u + v,\ a_1 = 2u - v = 7.$$

So $u = 3, v = -1$, therefore

$$a_n = 3 \cdot 2^n - (-1)^n.$$

Exercise 53. Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ be a linear homogeneous recurrence. Assume both sequences a_n, a'_n satisfy this linear homogeneous recurrence. Show that $a_n + a'_n$ and αa_n also satisfy it, for α some constant.

Solution. We have

$$\begin{aligned} a_n + a'_n &= (c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}) + (c_1 a'_{n-1} + c_2 a'_{n-2} + \ldots + c_k a'_{n-k}) \\ &= c_1(a_{n-1} + a'_{n-1}) + c_2(a_{n-2} + a'_{n-2}) + \ldots + c_k(a_{n-k} + a'_{n-k}). \end{aligned}$$

Thus $a_n + a'_n$ is a solution of the recurrence. Similarly

$$\begin{aligned} \alpha a_n &= \alpha(c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}) \\ &= c_1 \alpha a_{n-1} + c_2 \alpha a_{n-2} + \ldots + c_k \alpha a_{n-k}. \end{aligned}$$

Therefore αa_n is a solution of the recurrence.

Exercises for Chapter 7

Exercise 54. Set $i = \sqrt{-1}$. Compute

$$i^5, \ \frac{1}{i^2}, \ \frac{1}{i^3}.$$

Solution. First

$$i^5 = (i^4)i = i.$$

Note that $i^4 = 1$ because $i^2 = -1$.

Then since $i^2 = -1$, we have

$$\frac{1}{i^2} = -1.$$

Finally since $i^3 = -i$, we have

$$\frac{1}{i^3} = \frac{1}{-i} = i.$$

Indeed $-i \cdot i = 1$.

Exercise 55. Set $i = \sqrt{-1}$. Compute the real part and the imaginary part of

$$\frac{(1+2i)-(2+i)}{(2-i)(3+i)}.$$

Solution. We have

$$\begin{aligned}\frac{(1+2i)-(2+i)}{(2-i)(3+i)} &= \frac{-1+i}{7-i} \\ &= \frac{(-1+i)(7+i)}{(7-i)(7+i)} \\ &= \frac{-4}{25} + i\frac{3}{25}.\end{aligned}$$

Exercise 56. Set $i = \sqrt{-1}$. Compute $d, e \in \mathbb{R}$ such that

$$4 - 6i + d = \frac{7}{i} + ei.$$

Solution. Note that for a complex number to be zero, we need both its real and its imaginary parts to be 0. We thus need $4 + d = 0$, that is $d = -4$, and

$$\frac{7}{i} + ei + 6i = 0,$$

for which we need $e = 1$.

Exercise 57. For $z_1, z_2 \in \mathbb{C}$, prove that

- $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$.
- $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$.

Solution. Write $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$.

- We have

$$\begin{aligned}\overline{z_1 + z_2} &= \overline{(a_1 + a_2) + i(b_1 + b_2)} \\ &= (a_1 + a_2) - i(b_1 + b_2) \\ &= a_1 - ib_1 + a_2 - ib_2 \\ &= \bar{z}_1 + \bar{z}_2.\end{aligned}$$

- Similarly:

$$\begin{aligned}\overline{z_1 \cdot z_2} &= \overline{(a_1 + ib_1)(a_2 + ib_2)} \\ &= a_1a_2 - i(a_1b_2 + b_1a_2) - b_1b_2 \\ &= (a_1 - ib_1)(a_2 - ib_2) \\ &= \bar{z}_1 \cdot \bar{z}_2.\end{aligned}$$

Exercise 58. Consider the complex number z in polar form: $z = re^{i\theta}$. Express $re^{-i\theta}$ as a function of z.

Solution. Since

$$z = r(\cos\theta + i\sin\theta),$$

we have

$$re^{-i\theta} = r(\cos\theta - i\sin\theta) = \bar{z}.$$

Exercise 59. Prove that

$$(\cos x + i\sin x)^n = \cos nx + i\sin nx$$

for n an integer.

Solution. We have by Euler formula that

$$e^{ix} = \cos x + i\sin x,$$

thus

$$(e^{ix})^n = e^{inx}\cos nx + i\sin nx.$$

Exercise 60. Compute $|e^{i\theta}|$, $\theta \in \mathbb{R}$.

Solution. We have

$$|e^{i\theta}|^2 = |\cos\theta + i\sin\theta|^2 = (\cos\theta)^2 + (\sin\theta)^2 = 1.$$

Therefore $|e^{i\theta}| = 1$.

Exercise 61. Prove the so-called triangle inequality:

$$|a + b| \leq |a| + |b|,\ a, b \in \mathbb{C}.$$

Solution. Write $a = a_1 + ia_2$, $b = b_1 + ib_2$. Then

$$|a+b|^2 = (a+b)\overline{(a+b)} = |a|^2 + |b|^2 + a\bar{b} + \bar{a}b$$

while

$$(|a| + |b|)^2 = |a|^2 + 2|a||b| + |b|^2$$

so we are left to show that $a\bar{b} + \bar{a}b \leq 2|a||b|$. We have

$$a\bar{b} = (a_1 + ia_2)\overline{(b_1 + ib_2)} = a_1b_1 - a_1ib_2 + ia_2b_1 + a_2b_2$$

and

$$\bar{a}b = \overline{(a_1 + ia_2)}(b_1 + ib_2) = a_1b_1 + a_1ib_2 - ia_2b_1 + a_2b_2$$

thus we are left to show that

$$a\bar{b} + \bar{a}b = 2a_1b_1 + 2a_2b_2 \leq 2|a||b|,$$

or equivalently that $(a_1b_1 + a_2b_2)^2 \leq |ab|^2$. But $|ab|^2 = (a_1^2 + a_2^2)(b_1^2 + b_2^2)$ and we get

$$2a_1b_1a_2b_2 \leq a_1^2b_2^2 + a_2^2b_1^2 \iff 0 \leq (a_1b_2 - a_2b_1)^2$$

which is true.

Exercise 62. Compute the two roots of $4i$, that is

$$\sqrt{4i}.$$

Solution. We have

$$\sqrt{4i} = 2\sqrt{i} = \pm 2e^{2\pi i/8} = \pm 2(\cos 2\pi/8 + i \sin 2\pi/8).$$

One may further compute that $\cos 2\pi/8 = \sin 2\pi/8 = 1/\sqrt{2}$ (this is for example seen by considering a square with edges of size 1). Therefore the two roots are

$$\pm 2(1/\sqrt{2} + i/\sqrt{2}).$$

Exercises for Chapter 8

Exercise 63. Compute the sum $A + B$ of the matrices A and B, where A and B are as follows:

1.
$$A = \begin{pmatrix} 2 & \sqrt{2} \\ -1 & 3 \end{pmatrix}, \ B = \begin{pmatrix} 0 & \sqrt{2} \\ 4 & 2 \end{pmatrix}$$
where A, B are matrices with coefficients in $\mathbb{R}$.

2.
$$A = \begin{pmatrix} 2+i & -1 \\ -1+i & 3 \end{pmatrix}, \ B = \begin{pmatrix} -i & 1 \\ -1 & 2 \end{pmatrix}$$
where A, B are matrices with coefficients in $\mathbb{C}$, and $i = \sqrt{-1}$.

3.
$$A = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 2 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$
where A, B are matrices with coefficients that are integers $\mod 3$.

What are the dimensions of the matrices involved?

Solution. 1. Matrices are 2×2.
$$A + B = \begin{pmatrix} 2 & \sqrt{2} \\ -1 & 3 \end{pmatrix} + \begin{pmatrix} 0 & \sqrt{2} \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2\sqrt{2} \\ 3 & 5 \end{pmatrix}$$

2. Matrices are 2×2.
$$A + B = \begin{pmatrix} 2+i & -1 \\ -1+i & 3 \end{pmatrix} + \begin{pmatrix} -i & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -2+i & 5 \end{pmatrix}$$

3. Matrices are 2×3.
$$A + B = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

Exercise 64. 1. Compute the transpose A^T of A for
$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}.$$

2. Show that $(A + B)^T = A^T + B^T$.

Solution. 1. We have

$$A^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{pmatrix}.$$

2. If A has coefficients $a_{i,j}$, then the coefficients of A^T are $a_{j,i}$, thus $(A+B)^T$ has coefficients $a_{j,i} + b_{j,i}$ and $(A+B)^T = A^T + B^T$ as needed.

Exercise 65. Compute

$$2A + BC + B^2 + AD$$

where

$$A = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}, C = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$$

are real matrices and $D = I_2$ is the 2-dimensional identity matrix.

Solution. First we notice that $AD = A$, thus $2A + AD = 3A$, and we are left to compute

$$3A + BC + B^2.$$

Then

$$3A = \begin{pmatrix} 6 & 0 \\ -3 & 3 \end{pmatrix},$$

and

$$BC = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -3 & -1 \\ 4 & 2 \end{pmatrix}$$

and finally

$$B^2 = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 0 & 4 \end{pmatrix}$$

therefore

$$3A + BC + B^2 = \begin{pmatrix} 6 & 0 \\ -3 & 3 \end{pmatrix} + \begin{pmatrix} -3 & -1 \\ 4 & 2 \end{pmatrix} + \begin{pmatrix} 1 & -3 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 4 & -4 \\ 1 & 9 \end{pmatrix}$$

Exercise 66. Consider the complex matrix

$$A = \begin{pmatrix} 0 & i \\ -1 & 0 \end{pmatrix},$$

where $i = \sqrt{-1}$. What is A^l, for $l \geq 1$.

Solution. We have

$$A^2 = \begin{pmatrix} 0 & i \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} = -iI_2$$

where I_2 is the identity matrix. Thus

$$A^3 = -i \begin{pmatrix} 0 & i \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix}$$

and

$$A^4 = (A^2)^2 = (-iI_2)^2 = -I_2.$$

Next

$$A^5 = -A,\ A^6 = -A^2 = iI_2,\ A^7 = -A^3 = iA,\ A^8 = (A^4)^2 = I_2.$$

This shows that A^l is decided by $l \mod 8$, since we may write $l = l' + 8k$ for some integers l', k, and

$$A^l = A^{l'+8k} = A^{l'}(A^8)^k = A^{l'}$$

so we conclude that

$$A^l = \begin{cases} A & l \equiv 1 \mod 8 \\ -iI_2 & l \equiv 2 \mod 8 \\ -iA & l \equiv 3 \mod 8 \\ -I_2 & l \equiv 4 \mod 8 \\ -A & l \equiv 5 \mod 8 \\ iI_2 & l \equiv 6 \mod 8 \\ iA & l \equiv 7 \mod 8 \\ I_2 & l \equiv 0 \mod 8 \end{cases}$$

Exercise 67. 1. Let S be the set of 3×3 diagonal real matrices. Is S closed under matrix addition?

2. Consider the real matrix

$$A = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}.$$

Compute a matrix B such that $A + B$ is a diagonal, and a matrix C such that AC is diagonal.

Solution. 1. Take two diagonal matrices D and E in S

$$D = \begin{pmatrix} d_{1,1} & 0 & 0 \\ 0 & d_{2,2} & 0 \\ 0 & 0 & d_{3,3} \end{pmatrix}, \; E = \begin{pmatrix} e_{1,1} & 0 & 0 \\ 0 & e_{2,2} & 0 \\ 0 & 0 & e_{3,3} \end{pmatrix}$$

and compute their sum:

$$D + E = \begin{pmatrix} d_{1,1} + e_{1,1} & 0 & 0 \\ 0 & d_{2,2} + e_{2,2} & 0 \\ 0 & 0 & d_{3,3} + e_{3,3} \end{pmatrix}$$

thus the sum $D+E$ belongs to S and S is closed under matrix addition.

2. We want a matrix B such that

$$A + B = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} = \begin{pmatrix} 2 + b_{1,1} & 3 + b_{1,2} \\ -1 + b_{2,1} & 1 + b_{2,2} \end{pmatrix}$$

is diagonal. Thefore we need $b_{1,2} = -3$ and $b_{2,1} = 1$, then such a matrix B will work, independently of the choice of $b_{1,1}$ and $b_{2,2}$. Then we want a matrix C such that

$$AC = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix} = \begin{pmatrix} 2c_{1,1} + 3c_{2,1} & 2c_{1,2} + 3c_{2,2} \\ -c_{1,1} + c_{2,1} & -c_{1,2} + c_{2,2} \end{pmatrix}$$

is diagonal. For example, we can take $c_{1,2} = -3$ and $c_{2,2} = 2$, and $c_{2,1} = c_{1,1}$.

Exercise 68. Let A and B be $n \times n$ matrices which satisfy

$$A^2 + AB + A - I_n = 0,$$

where I_n means the $n \times n$ identity matrix, and 0 the $n \times n$ zero matrix. Show that A is invertible.

Solution. We can rewrite $A^2 + AB + A - I_n = 0$ as

$$A(A + B + I_n) = I_n$$

therefore A is invertible with inverse $A + B + I_n$.

Exercise 69. Compute, if it exists, the inverse A^{-1} of the matrix A, where A is given by

- $$A = \begin{pmatrix} 2 & 3 & -2 \\ -1 & 1 & 2 \\ 3 & 7 & 2 \end{pmatrix}$$

  for A a real matrix.

- $$A = \begin{pmatrix} 1 & 1+i \\ 1-i & 1 \end{pmatrix}$$

  for A a complex matrix and $i = \sqrt{-1}$.

- $$A = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$$

  for A a matrix with coefficients modulo 5.

Solution.

- We compute the row echelon form of the augmented matrix

  $$A = \begin{pmatrix} 2 & 3 & -2 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 3 & 7 & 2 & 0 & 0 & 1 \end{pmatrix}.$$

  Replace the 3rd row by (row 3)+3(row 2)=$(3,7,2,0,0,1)$+3$(-1,1,2,0,1,0)$ which is$(3,7,2,0,0,1)$+$(-3,3,6,0,3,0)$=$(0,10,8,0,3,1)$, to get

  $$\begin{pmatrix} 2 & 3 & -2 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 10 & 8 & 0 & 3 & 1 \end{pmatrix}.$$

  Next replace the 1st row by (row 1)+2(row 2)=$(2,3,-2,1,0,0)$+2$(-1,1,2,0,1,0)$ which is$(2,3,-2,1,0,0)$+$(-2,2,4,0,2,0)$=$(0,5,2,1,2,0)$, after which we switch row 1 and 2 to get

  $$\begin{pmatrix} -1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 5 & 2 & 1 & 2 & 0 \\ 0 & 10 & 8 & 0 & 3 & 1 \end{pmatrix}.$$

Next replace the 3rd row by -2(row 2)+(row 3)=$(0, -10, -4, -2, -4, 0)$+$(0, 10, 8, 0, 3, 1)$ which is$(0, 0, 4, -2, -1, 1)$, to get

$$\begin{pmatrix} -1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 5 & 2 & 1 & 2 & 0 \\ 0 & 0 & 4 & -2 & -1 & 1 \end{pmatrix}.$$

Therefore we can already tell that this matrix is invertible. Next to find its inverse, let us compute the reduced row echelon form. Replace row 2 by 2(row 2)-(row 3)=$(0, 10, 4, 2, 4, 0)$-$(0, 0, 4, -2, -1, 1)$ which is $(0, 10, 0, 4, 5, -1)$. We get

$$\begin{pmatrix} -1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 10 & 0 & 4 & 5 & -1 \\ 0 & 0 & 4 & -2 & -1 & 1 \end{pmatrix}.$$

Next replace (row 1) by 2(row 1)-(row 3)=$(-2, 2, 4, 0, 2, 0)$+$(0, 0, -4, 2, 1, -1)$=$(-2, 2, 0, 2, 3, -1)$ to get

$$\begin{pmatrix} -2 & 2 & 0 & 2 & 3 & -1 \\ 0 & 10 & 0 & 4 & 5 & -1 \\ 0 & 0 & 4 & -2 & -1 & 1 \end{pmatrix}.$$

Next we replace (row 1) by -5(row 1)+(row 2)=$(10, -10, 0, -10, -15, 5)$+$(0, 10, 0, 4, 5, -1)$ which is equal to $(10, 0, 0, -6, -10, 4)$

$$\begin{pmatrix} 10 & 0 & 0 & -6 & -10 & 4 \\ 0 & 10 & 0 & 4 & 5 & -1 \\ 0 & 0 & 4 & -2 & -1 & 1 \end{pmatrix}.$$

We are now left by normalizing the diagonal coefficients to get 1:

$$\begin{pmatrix} 1 & 0 & 0 & -6/10 & -1 & 4/10 \\ 0 & 1 & 0 & 4/10 & 5/10 & -1/10 \\ 0 & 0 & 1 & -1/2 & -1/4 & 1/4 \end{pmatrix}.$$

This gives us A^{-1}:

$$A^{-1} = \begin{pmatrix} -6/10 & -1 & 4/10 \\ 4/10 & 5/10 & -1/10 \\ -1/2 & -1/4 & 1/4 \end{pmatrix}.$$

It is always good to compute AA^{-1} to make sure the answer is correct!

- We compute the row echelon form of
$$A = \begin{pmatrix} 1 & 1+i & 1 & 0 \\ 1-i & 1 & 0 & 1 \end{pmatrix}.$$
We replace (row 2) by (row 2)-$(1-i)$(row 1)=$(1-i,1,0,1)$-$(1-i,2,1-i,0)$=$(0,-1,-1+i,1)$, to get
$$\begin{pmatrix} 1 & 1+i & 1 & 0 \\ 0 & -1 & -1+i & 1 \end{pmatrix}.$$
Thus this matrix is invertible, and we compute its reduced row echelon form. We replace (row 1) by (row 1)+$(1+i)$(row 2)=$(1,1+i,1,0)$+$(0,-(1+i),-(1+i)(1-i),1+i)$ to get
$$\begin{pmatrix} 1 & 0 & -1 & 1+i \\ 0 & -1 & -1+i & 1 \end{pmatrix}.$$
Finally we multiply the second row by -1:
$$\begin{pmatrix} 1 & 0 & -1 & 1+i \\ 0 & 1 & 1-i & -1 \end{pmatrix}.$$
This gives
$$A^{-1} = \begin{pmatrix} -1 & 1+i \\ 1-i & -1 \end{pmatrix}.$$
- We compute the row echelon form of
$$A = \begin{pmatrix} 2 & 3 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$
We replace (row 1) by (row 1)-2(row 2)=$(2,3,1,0)$-2$(1,1,0,1)$=$(0,1,1,-2)$, and switch rows to get
$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 \end{pmatrix}.$$
This matrix is thus invertible, and we compute its reduced row echelon form. Replace row 1 by (row 1)-(row 2)=$(1,0,-1,-2)$ to get
$$A = \begin{pmatrix} 1 & 0 & 4 & 3 \\ 0 & 1 & 1 & 3 \end{pmatrix}.$$
Therefore
$$A^{-1} = \begin{pmatrix} 4 & 3 \\ 1 & 3 \end{pmatrix}.$$

Exercise 70. Write the following system of linear equations in a matrix form and solve it.

$$\begin{cases} x_1 + x_2 - 2x_3 &= 1 \\ 2x_1 - 3x_2 + x_3 &= -8 \\ 3x_1 + x_2 + 4x_3 &= 7 \end{cases}$$

Solution. In matrix form, we get

$$\begin{pmatrix} 1 & 1 & -2 \\ 2 & -3 & 1 \\ 3 & 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -8 \\ 7 \end{pmatrix}.$$

We compute the row echelon form of the matrix of the system, augmented by the vector $(1, -8, 7)$: Replace row 2 by (row 2)-2(row 1) and row 3 by (row 3)-3(row 1):

$$\begin{pmatrix} 1 & 1 & -2 & 1 \\ 0 & -5 & 5 & -10 \\ 0 & -2 & 10 & 4 \end{pmatrix}$$

Replace row 3 by 5(row 3)-2(row 2):

$$\begin{pmatrix} 1 & 1 & -2 & 1 \\ 0 & -5 & 5 & -10 \\ 0 & 0 & 40 & 40 \end{pmatrix}$$

Divide the last row by 40, we already deduce that $x_3 = 1$, and after dividing the second row by -5, we get

$$\begin{pmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

where we replace row 2 by (row 2)+(row 3):

$$\begin{pmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

from which we get that $x_2 = 3$. Finally, by replacing row 1 by (row 1)+2(row 3), and then (row 1) again by (row 1)-(row 2) we get

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

and $x_1 = 0$. In summary, this system has a unique solution, given by

$$x_1 = 0,\ x_2 = 3,\ x_3 = 1.$$

Exercise 71. Write the following system of linear equations in a matrix form and solve it.

$$\begin{cases} x_1 - x_2 + x_3 - x_4 & = & 2 \\ x_1 - x_2 + x_3 + x_4 & = & 0 \\ 4x_1 - 4x_2 + 4x_3 & = & 4 \\ -2x_1 + 2x_2 - 2x_3 + x_4 & = & -3 \end{cases}$$

Solution. In matrix form, we have

$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 4 & -4 & 4 & 0 \\ -2 & 2 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 4 \\ -3 \end{pmatrix}.$$

Next we compute the row echelon form of

$$\begin{pmatrix} 1 & -1 & 1 & -1 & 2 \\ 1 & -1 & 1 & 1 & 0 \\ 4 & -4 & 4 & 0 & 4 \\ -2 & 2 & -2 & 1 & -3 \end{pmatrix}$$

Replace (row 4) by (row 4)+2(row 1) and (row 3) by (row 3)-4(row 1):

$$\begin{pmatrix} 1 & -1 & 1 & -1 & 2 \\ 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

Then row 3 is a multiple of row 4, and replace row 2 by (row 2)-(row 1):

$$\begin{pmatrix} 1 & -1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

showing the row 2 is a multiple of row 4, and thus our system reduces to

$$\begin{pmatrix} 1 & -1 & 1 & -1 & 2 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

where we replace row 1 by (row 1)-(row 2):

$$\begin{pmatrix} 1 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

This tells us that

$$x_4 = -1, \; x_1 = 1 + x_2 - x_3.$$

There are infinitely many solutions.

Exercises for Chapter 9

Exercise 72. Consider the sets $A = \{1, 2\}$, $B = \{1, 2, 3\}$ and the relation $(x, y) \in R \iff (x - y)$ is even. Compute the inverse relation R^{-1}. Compute its matrix representation.

Solution. The relation R is

$$(1, 1), (1, 3), (2, 2),$$

therefore the relation R^{-1} is

$$(1, 1), (3, 1), (2, 2).$$

Its matrix representation is obtained by representing B as rows, that is row 1 is $b_1 = 1$, row 2 is $b_2 = 2$, row 3 is $b_3 = 3$, while column 1 is $a_1 = 1$ and column 2 is $a_2 = 2$:

$$\begin{pmatrix} T & F \\ F & T \\ T & F \end{pmatrix}$$

Exercise 73. Consider the sets $A = \{2, 3, 4\}$, $B = \{2, 6, 8\}$ and the relation $(x, y) \in R \iff x \mid y$. Compute the matrix of the inverse relation R^{-1}.

Solution. The relation R is

$$(2, 2), (2, 6), (2, 8), (3, 6), (4, 8)$$

thus the inverse relation R^{-1} is

$$(2, 2), (6, 2), (8, 2), (6, 3), (8, 4)$$

that is $(x,y) \in R^{-1} \iff x$ is a multiple of y, and the corresponding matrix is

$$\begin{pmatrix} T & F & F \\ T & T & F \\ T & F & T \end{pmatrix}$$

Exercise 74. Let R be a relation from $\mathbb{Z}$ to $\mathbb{Z}$ defined by $xRy \leftrightarrow 2|(x-y)$. Show that if n is odd, then n is related to 1.

Solution. Any odd number n can be written of the form $n = 2m+1$ for some integer m. Therefore $n-1 = 2m$ which is divisible by 2 and n is related to 1.

Exercise 75. This exercise is about composing relations.

1. Consider the sets $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, $C = \{c_1, c_2, c_3\}$ with the following relations R from A to B, and S from B to C:

   $$R = \{(a_1,b_1),(a_1,b_2)\},\ S = \{(b_1,c_1),(b_2,c_1),(b_1,c_3),(b_2,c_2)\}.$$

   What is the matrix of $R \circ S$?

2. In general, what is the matrix of $R \circ S$?

Solution. 1. Let us write the matrices of R and S first:

$$\begin{pmatrix} T & T \\ F & F \end{pmatrix} \circ \begin{pmatrix} T & F & T \\ T & T & F \end{pmatrix}.$$

Next we have that $(a,c) \in R \circ S$ whenever $aRb \wedge bSc$ for some $b \in B$. So to know, for example, whether (a_1, c_1) is in $R \circ S$, we have to check if we can find a b_i such that $(a_1, b_i) \wedge (b_i, c_1)$, that is whether

$$[(a_1,b_1) \wedge (b_1,c_1)] \vee [(a_1,b_2) \wedge (b_2,c_1)]$$

is true. But (a_i, b_j) means that the coefficient r_{ij} of the matrix R is true, and similarly (b_i, a_j) means that the coefficient s_{ij} of the matrix S is true. So we may rephrase the coefficient of the 1st row, 1st column of the matrix of $R \circ S$ as

$$(r_{11} \wedge s_{11}) \vee (r_{12} \wedge s_{21}).$$

Notice that this is almost like doing the scalar product of the first row of R with the first column of S, except that multiplication is replaced by $\wedge$, and addition by $\vee$. Therefore, we have that the matrix of $R \circ S$ is

$$\begin{pmatrix} T & T \\ F & F \end{pmatrix} \circ \begin{pmatrix} T & F & T \\ T & T & F \end{pmatrix} = \begin{pmatrix} T & T & T \\ F & F & F \end{pmatrix}$$

2. In general, we have a relation R from A to B, and a relation S from B to C, where the size of the set B is n. Denote the coefficients of the matrix of the relation R by r_{ij}, and that of the matrix of the relation S by s_{ij}. Then the matrix of $R \circ S$ will have coefficients t_{ij} given by

$$t_{ij} = (r_{i1} \wedge s_{1j}) \vee (r_{i2} \wedge s_{2j}) \vee \ldots \vee (r_{in} \wedge s_{nj}).$$

Exercise 76. Consider the relation R on $\mathbb{Z}$, given by $aRb \iff a - b$ divisible by n. Is it symmetric?

Solution. Yes it is symmetric. Suppose aRb, then $a - b$ is divisible by n. Thus $-(a - b) = b - a$ is divisible by n, and bRa holds.

Exercise 77. Consider a relation R on any set A. Show that R symmetric if and only if $R = R^{-1}$.

Solution. Consider a relation R. The relation R^{-1} is defined by pairs (y, x) such that $(x, y) \in R$. If R is symmetric, it has the property that $(x, y) \Rightarrow (y, x)$, therefore $(y, x) \in R$ and $R = R^{-1}$. Conversely, if $R = R^{-1}$, then if $(x, y) \in R$, it must be that $(y, x) \in R$ and R is symmetric.

Exercise 78. Consider the set $A = \{a, b, c, d\}$ and the relation

$$R = \{(a, a), (a, b), (a, d), (b, a), (b, b), (c, c), (d, a), (d, d)\}.$$

Is this relation reflexive? symmetric? transitive?

Solution. It is reflexive since $(a, a), (b, b), (c, c), (d, d) \in R$. It is symmetric since $(a, b), (b, a)$, $(a, d), (d, a) \in R$. It is not transitive, indeed, $(b, a), (a, d) \in R$ but $(b, d) \notin R$.

Exercise 79. Consider the set $A = \{0, 1, 2\}$ and the relation $R = \{(0, 2), (1, 2), (2, 0)\}$. Is R antisymmetric?

Solution. No, since $(0, 2)$ and $(2, 0)$ are in R, but $2 \neq 0$.

Exercise 80. Are symmetry and antisymmetry mutually exclusive?

Solution. There is no connection between symmetry and antisymmetry, so no they are not mutually exclusive. For example, the relation $A = B$ is both symmetric and antisymmetric. Then the relation "A is brother of B" is neither symmetric (if A is a brother of B, it could be that B is a sister of A) nor antisymmetric.

Exercise 81. Consider the relation R given by divisibility on positive integers, that is $xRy \leftrightarrow x|y$. Is this relation reflexive? symmetric? antisymmetric? transitive? What if the relation R is now defined over non-zero integers instead?

Solution. It is reflexive since $x|x$ always. It is not symmetric, since for example $1|y$ but y will never divide 1 if $y > 1$. It is antisymmetric, since if $x|y$ then $y = ax$ while if $y|x$ then $x = by$ and it must be that $y = ax = a(by) = aby$ and $a = b = 1$. It is transitive, since $x|y$ and $y|z$ imply $y = ax$, $z = by$ thus $z = by = b(ax)$ and $x|z$.

If we consider instead non-zero integers, the relation is not antisymmetric, indeed $y = ax = a(by) = aby$ could imply $a = b = -1$ in which case $x|y$ and $y|x$ when $y = -x$ is possible.

Exercise 82. Consider the set $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. Show that the relation $xRy \leftrightarrow 2|(x - y)$ is an equivalence relation.

Solution. It is reflexive: $2|(x-x)$. It is symmetric: if $2|(x-y)$ then $(x-y) = 2n$ for some integer n, and thus $(y - x) = -2n$ showing that $2|(y - x)$. It is transitive: if $2|(x-y)$ and $2|(y-z)$, then $(x-y) = 2n$, and $(y-z) = 2m$, for some integers m, n. Therefore $x-z = (x-y)+(y-z) = 2n+2m = 2(n+m)$ and $2|(x - z)$.

Exercise 83. Show that given a set A and an equivalence relation R on A, then the equivalence classes of R partition A.

Solution. Let $a, b \in A$, and $[a], [b]$ denote their equivalence classes. It is possible that $[a] = [b]$. Suppose that this is not the case. Then we will show that $[a]$ and $[b]$ are disjoint. Suppose by contradiction that there exists one element $c \in [a] \cap [b]$. Then aRc and bRc. But R is an equivalence relation, therefore it is symmetric (and cRb) and transitive, implying that aRb. But then $b \in [a]$ and $a \in [b]$ by symmetry, and it must be that $[a] = [b]$. Indeed:

we show that $[a] \subseteq [b]$ and $[b] \subseteq [a]$. Take an element x of $[a]$, then aRx, that is xRa (symmetry), and since aRb (b is in $[a]$), it must be that xRb (transitivity) and thus bRx (symmetry again), which shows that x is in $[b]$. The same reasoning will show that $[b]$ belongs to $[a]$.

Since either $[a] = [b]$ or they are disjoint, take the union of the classes $[a]$ that give distinct classes, and this gives a partition of A.

Exercise 84. Consider the set $A = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and the relation

$$xRy \leftrightarrow \exists c \in \mathbb{Z},\ y = cx.$$

Is R an equivalence relation? is R a partial order?

Solution. R is reflexive: $xRx \leftrightarrow \exists c \in \mathbb{Z},\ x = cx$, take $c = 1$. R is not symmetric: xRy means $x = cy$, but then $y = \frac{x}{c}$ so apart if $c = \pm 1$, $\frac{1}{c}$ will not be in $\mathbb{Z}$. For example, $2R4$ since $4 = c2$ with $c = 2$, but $2 = c4$ means that c cannot be an integer. We conclude that R cannot be an equivalence relation.

Let us check antisymmetry and transitivity. Suppose $y = cx$ and $x = c'y$, then $x = c'cx$ and $c'c = 1$. So either $c = c' = -1$, which cannot happen because all elements of A are positive, or $c = c' = 1$, and the relation is antisymmetric. For transitivity, suppose $xRy \iff y = cx$, $yRz \iff z = c'y$. Then $z = c'y = c'cx$ with $cc' \in \mathbb{Z}$ thus xRz as needed. We conclude that R is a partial order.

Exercises for Chapter 10

Exercise 85. Consider the set $A = \{a, b, c\}$ with power set $P(A)$ and $\cap$: $P(A) \times P(A) \to P(A)$. What is its domain? its co-domain? its range? What is the cardinality of the pre-image of $\{a\}$?

Solution. Its domain is the cartesian product $P(A) \times P(A)$, its co-domain is $P(A)$. Its range is $P(A)$: indeed, for any subset X of A, $X \cap X = X$, therefore every element of $P(A)$ has a pre-image. The pre-image of $\{a\}$ is the set of elements in $P(A) \times P(A)$ which are mapped to $\{a\}$, that is, pairs (X, Y) of subsets of A whose intersection is $\{a\}$. Now $\{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}$ are all the subsets containing $\{a\}$, so this gives 2^4 possible pairs, but among them, not all are suitable: we have to remove those with bigger intersection. So we can intersect $\{a\}$ with all of them:

$$(\{a\}, \{a\}),\ (\{a\}, \{a, b\}),\ (\{a\}, \{a, c\}),\ (\{a\}, \{a, b, c\}),$$

or $\{a, c\}$ with $\{a, b\}$. Note that the ordering of a pair matters, thus all those pairs give rise to another pair, apart for $(\{a\}, \{a\})$ thus a total of 9.

Exercise 86. Show that $\sin : \mathbb{R} \to \mathbb{R}$ is not one-to-one.

Solution. We have that $\sin(0) = \sin(\pi) = 0$ but $\pi \neq 0$, which contradicts the definition of one-to-one, since there exist $x_1 = 0, x_2 = \pi$ such that $\sin(x_1) = \sin(x_2)$ but $x_1 \neq x_2$.

Exercise 87. Show that $\sin : \mathbb{R} \to \mathbb{R}$ is not onto, but $\sin : \mathbb{R} \to [-1, 1]$ is.

Solution. It is not onto because $\exists y \in \mathbb{R}$, say $y = 2$, such that for all $x \in \mathbb{R}$, $f(x) \neq 2$.

Exercise 88. Is $h : \mathbb{Z} \to \mathbb{Z}$, $h(n) = 4n - 1$, onto (surjective)?

Solution. No, it is not. For example, take $y = 1$. Then it is not possible that $1 = 4n - 1$ for n an integer, because this equation means that $n = 1/2$.

Exercise 89. Is $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$, a bijection (one-to-one correspondence)?

Solution. Injectivity: suppose $f(x_1) = f(x_2)$, then $x_1^3 = x_2^3$ and it must be that $x_1 = x_2$. Surjectivity: take $y \in \mathbb{R}$, and $x = \sqrt[3]{y} \in \mathbb{R}$, then $f(x) = y$, so surjectivity holds. Therefore it is a bijection.

Exercise 90. Consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ and $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x + 5$. What is $g \circ f$? What is $f \circ g$?

Solution. We have

$$g(f(x)) = g(x^2) = x^2 + 5, \; f(g(x)) = f(x + 5) = (x + 5)^2.$$

Exercise 91. Consider $f : \mathbb{Z} \to \mathbb{Z}$, $f(n) = n + 1$ and $g : \mathbb{Z} \to \mathbb{Z}$, $g(n) = n^2$. What is $g \circ f$? What is $f \circ g$?

Solution. We have

$$g(f(n)) = g(n + 1) = (n + 1)^2, \; f(g(n)) = f(n^2) = n^2 + 1.$$

Exercise 92. Given two functions $f : X \to Y$, $g : Y \to Z$. If $g \circ f : X \to Z$ is one-to-one, must both f and g be one-to-one? Prove or give a counter-example.

Solution. It is not true. For example, take $f : X \to Y$ and $g : Y \to Z$ as follows, $X = \{a, b, c\}$, $Y = \{w, x, y, z\}$, $Z = \{1, 2, 3\}$:

$$f(a) = x, \; f(b) = y, \; f(c) = z, \; g(w) = 1, \; g(x) = 1, \; g(y) = 2, \; g(z) = 3.$$

Then $g \circ f$ is one-to-one, but g is not.

Exercise 93. Show that if $f : X \to Y$ is invertible with inverse function $f^{-1} : Y \to X$, then $f^{-1} \circ f = i_X$ and $f \circ f^{-1} = i_Y$.

Solution. Take $x \in X$, with $y = f(x)$. Then $f^{-1}(f(x)) = f^{-1}(y) = x$ by definition of inverse, and $x = i_X(x)$ for all $x \in X$ therefore $f^{-1} \circ f = i_X$. Similarly take $y \in Y$ and $x = f^{-1}(y)$. Then $f(f^{-1}(y)) = f(x) = y$ by definition of inverse, and $y = i_Y(y)$ for all $y \in Y$ therefore $f \circ f^{-1} = i_Y$.

Exercise 94. If you pick five cards from a deck of 52 cards, prove that at least two will be of the same suit.

Solution. If you pick 5 cards, then the first one will be of a given suit (say heart), if the second is also heart, then you got two of the same suit. If the second is not heart (say diamond), then take a 3rd. If it is either heart or diamond, then you got at least two of the same suit, if not, say it is club, pick a 4rth card. Again, if the 4rth card is heart, diamond of club, you got at least two of the same suit, if not, it must be that this 4rth card is spade.

But now all the 4 possible choices of suits are picked, so no matter what is the next suit of the card, it will be one that has already appeared. This shows that you will always get at least two cards of the same suit. This is an application of the pigeonhole principle: you have 4 suits, and 5 cards, therefore 2 cards must be of the same suit.

Exercise 95. If you have 10 black socks and 10 white socks, and you are picking socks randomly, you will only need to pick three to find a matching pair.

Solution. Pick the first sock, it is say white. Pick the second sock, if it white, then you got a matching pair. If not, pick a third one. But by now, you have already one white and one black sock, so no matter which is the color of the third one, ou will have a matching pair. This is an application of the pigeonhole principle: you have 2 colors, and 3 socks, therefore 2 socks must be of the same color.

Exercises for Chapter 11

Exercise 96. Prove that if a connected graph G has exactly two vertices which have odd degree, then it contains an Euler path.

Solution. Suppose that v and w are the vertices of G which have odd degrees, while all the other vertices have an even degree. Create a new graph G', formed by G, with one more edge e, which connects v and w. Every vertex in G' has even degree, so by the theorem on Euler cycles, there is an Euler cycle. Say this Euler cycle is

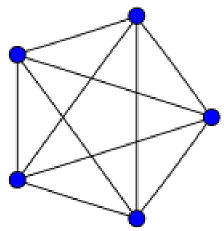
$$v, e_1, v_2, e_2, \ldots, w, e, v$$

then

$$v, e_1, v_2, e_2, \ldots, w$$

is an Euler path.

Exercise 97. Draw a complete graph with 5 vertices.



Solution.

Exercise 98. Show that in every graph G, the number of vertices of odd degree is even.

Solution. Let E denote the set of edges, and write the set V of vertices as $V' \cup V''$ where V' is the set of nodes with odd degrees, and V'' is the set of nodes with even degrees. Suppose that the number of vertices of odd degree is odd, then

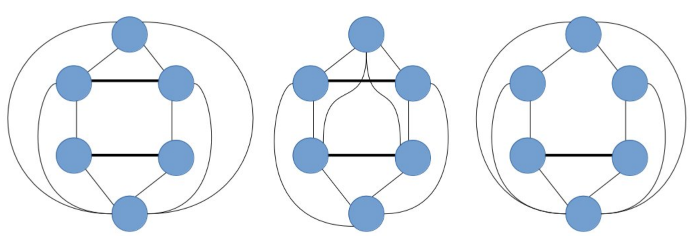
$$2|E| = \sum_{v \in V} \deg(v) = \sum_{v \in V'} + \sum_{v \in V''}$$

where the first sum (over V') is odd and the second sum is even, a contradiction.

Exercise 99. Show that in very simple graph (with at least two vertices), there must be two vertices that have the same degree.

Solution. Suppose there are n nodes. If all degrees are different, they must be exactly $0, 1, \ldots, n-1$, which is impossible: one cannot have one node of degree 0, yet another one with degree $n-1$!

Exercise 100. Decide whether the following graphs contain a Euler path/cycle.



Solution. The first graph (left hand side) contains a Euler path and no Euler circuit, the middle graph contains a Euler circuit, the third one contains none!

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